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Optimal Policies for Investment with Time-Varying Return Distributions

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We develop a model in which investors must learn the distribution of asset returns over time. The process of learning is made more difficult by the fact that the distributions are not constant through time. We consider risk-neutral investors who have quadratic utility and are selecting between two risky assets. We determine the time at which it is optimal to update the distribution estimate and hence, alter portfolio weights. Our results deliver an optimal policy for asset allocation, that is, the sequence of time intervals at which it is optimal to switch between assets, based on stochastic optimal control theory. In addition, we determine the time intervals in which asset switching leads to a loss with high probability. We provide estimates of the effectiveness of the optimal policy.

KEY WORDS: asset allocation; optimal policy; stochastic control; two-armed Bandit problem.

1. INTRODUCTION

Rational expectations has established a long tradition in economics. The theory of rational expectations has two components: first, that individuals maximize a specified function subject to constraints; and second, that the constraints perceived by individuals are mutually consistent. In applications to portfolio selection in finance, the second component typically means that the individuals know the underlying probability distributions for asset returns. As Sargent (1993) describes, because individuals are assumed to know the underlying distributions, modeling individual learning is not a central feature of a rational expectations equilibrium. Recently, a literature has developed that moves away from rational expectations by removing the second component. That is, individuals still maximize a specified function, but they do so in the presence of uncertainty about the

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underlying probability distribution. To resolve the uncertainty, the individuals learn about the underlying distribution in much the same way that the econometrician studying the problem does, by observing data and revising estimates of the data generating process. Much is now known about how agents learn in environments where the underlying distributions are unchanging through time. We study learning in an environment in which the underlying distributions are changing through time.

To study learning in a time-varying environment, we consider a risk-neutral investor who is selecting between two risky assets. The underlying distributions that generate the returns for the two assets change at unknown points in time. The investor thus faces two sources of uncertainty: first, the uncertainty about the distributions of the two assets in the first period; and second, the uncertainty that results from the changes in the distributions over time. Because the distributions change at unknown points in time, the problem is more complicated than typically arises in standard models of learning. We derive the optimal portfolio allocation strategy for the investor based on stochastic optimal control theory.

To understand how stochastic optimal control theory relates to the risk-neutral investor's portfolio allocation problem, consider the following model. Let the two assets be labeled asset 1 and asset 2 and assume that the returns on the assets are independent. Rather than modeling the returns to an asset directly, let the returns to an asset in a given period consist of the sum of cash payments (or flows) that occur over the period. Each payment in the cash flow stream is identical, both across time and assets, the variation in returns is introduced through variation in the frequency with which cash flows are paid. The cash flow stream for each of the assets is a Poisson process. It is known that the intensity of the underlying Poisson process can take on only two values, given by $\lambda > \mu$, and that the process underlying asset 1 has a different mean than the process underlying asset 2. Further, the returns process of the two assets switches at random points in time unknown to the investor. The points in time at which the mean cash flow from asset 1 switches between λ and μ are determined by a third independent Poisson process that has intensity a. Thus the investor's problem is dynamic; the investor needs to determine which asset has the underlying Poisson process with intensity λ at a given moment in time.

By modeling the cash flows generated by each asset, we are able to link directly to stochastic optimal control theory. The underlying data generating process for the cash flows corresponds to the two arms of the "two-armed bandit" that is studied in the stochastic control literature. Because the returns process switches at random points in time, the portfolio allocation problem corresponds to the two-armed bandit problem with unobserved switching between arms studied by Donchev (1995).

We describe the investor's problem with statistical hypotheses. The null hypothesis is that asset 1 has the lower mean return, μ, at a specific period. We denote this hypothesis H_1. At each moment in time the investor calculates the probability that H_1 is true. In period 0 the investor begins with one dollar and the *a priori*[3] probability x that H_1 is true. The investor's plan, or policy, consists of the investment allocation for each period. If we let the random variable U_t indicate the proportion of the dollar that is invested in asset 1 in period t, then the investor's policy is a random process $\pi = \{U_t\}_{t \geq 0}$ where for each period t, the random variable $0 \leq U_t \leq 1$ depends only on past observations of returns to the two assets. (We assume that the investor observes returns from both assets at each point in time.) We let $N_1(t)$ be the number of cash payments that accrue to the investor from asset 1 over the time span $[0, t]$ and let $N_2(t)$ be defined analogously for asset 2.

The number of cash payments that accrue to an investor over the period $(0, T)$ is

$$\int_0^T (dN_1(t) + dN_2(t)).$$

Because the number of cash payments is random, we work with the expected number of cash payments. By construction, the maximum value of the expected number of cash payments over the period $(0, T)$ is λT. The expected number of cash payments received by the investor depends both on the policy π and on x, which is the initial probability that asset 2 is the higher mean return asset. The expected difference between an investor's cash payments and the maximum number of cash payments, denoted $v^{\pi}(x)$, is

$$v^{\pi}(x) = E_x^{\pi} \int_0^{\infty} (dN_1(t) + dN_2(t) - \lambda\, dt) < 0, \tag{1.1}$$

where E_x^{π} denotes the expectation constructed under policy π with *a priori* probability x. (Presman and Sonin (1990) show that E_x^{π} is finite.) The policy φ is optimal if

$$v^{\varphi}(x) = \sup_{\pi} v^{\pi}(x). \tag{1.2}$$

The quantity $v^{\varphi}(x)$ corresponding to the optimal policy is said to be a value function of the problem. Let F_t denote all observations on $N_1(t)$ and $N_2(t)$ up to time t. The conditional, or *a posteriori*, probability that H_1 is true is $x_t = P_x(H_1|F_t)$, which depends on the *a priori* probability x.

[3] For notational simplicity, we fix the investment base at one dollar, that is the investor consumes each cash payment as soon as it is received.

In an interval of length T, a Poisson process with mean cash flow λ generates λT cash payments on average. The parameter λ is the jump rate of the Poisson process, which is the expected number of cash payments (or jumps) that accrue over one period. That is, if $EdN_1(t) = \lambda\, dt$, then λ is the jump rate for $N_1(t)$. Because $N_1(t)$ is the number of cash payments that flow to the investor from asset 1, the jump rate for $N_1(t)$ depends on both the jump rate for the asset and on the investor's perception about the quality of asset 1. (Clearly, if the investor thinks asset 1 is inferior to asset 2 over the entire interval and so invests nothing in asset 1, the jump rate for $N_1(t)$ is zero even if the underlying asset actually generated cash flows.) The investor's conditional expectation of the jump rate for $N_1(t)$ is $U_t[\mu x_t + \lambda(1 - x_t)]$, where U_t measures the proportion of the dollar invested in asset 1 and x_t is the investor's conditional probability that asset 1 has jump rate μ. Similarly, the investor's conditional expectation of the jump rate for $N_2(t)$ is $(1 - U_t)[\lambda x_t + \mu(1 - x_t)]$, so (1.1) is

$$v^{\pi}(x) = -E_x^{\pi} \int_0^{\infty} (\lambda - \mu)[U_t x_t + (1 - U_t)(1 - x_t)]\, dt. \tag{1.3}$$

The investor's decision problem is to choose π to maximize (1.3). Because there are two sources of uncertainty for the investor, the initial uncertainty about which asset has the higher expected return and the additional uncertainty from the "switching" of the higher expected return between assets, (1.3) is infinite for any reasonable policy. To ensure that the decision problem is well defined, we introduce a discount factor $\beta > 0$ so that the investor discounts future returns. The investor's decision problem is to choose π to maximize

$$v^{\pi}(x) = -E_x^{\pi} \int_0^{\infty} e^{-\beta t}[U_t x_t + (1 - U_t)(1 - x_t)]\, dt. \tag{1.4}$$

As we show below, to choose the optimal value of U_t, the investor only needs to know the *a posteriori* probability x_t. Thus, the optimal policy is a function $\varphi : [0, 1] \to [0, 1]$ that defines a correspondence between the values of the *a posteriori* probabilities and the optimal values of $\{U_t\}$. We let $U_t = 1$ correspond to the decision to invest the entire dollar in asset 1, $U_t = 0$ correspond to the decision to invest the entire dollar in asset 2, and the boundary case $U_t = \frac{1}{2}$ correspond to the decision to leave the entire dollar invested in the asset selected in the previous period. The optimal policy function is of the form

$$\varphi(x_t) = \begin{cases} 1, & x_t < \frac{1}{2} \\ 0, & x_t > \frac{1}{2} \\ \frac{1}{2}, & x_t = \frac{1}{2}. \end{cases} \tag{1.5}$$

The intuition for the result is straightforward. Because the investor is risk neutral, the investor chooses to maximize expected return without regard to risk. As a result, the investor will always choose to invest completely in the asset with the highest expected return, so the only relevant values for U_t are $(0, \frac{1}{2}, 1)$. Because $\varphi(\cdot)$ is a function only of x_t, the *a posteriori* probability x_t is a sufficient statistic for the optimal policy.

In Section 2 we describe the optimal policy that maximizes expected discounted returns. In Section 3 we outline the proof of our main result, namely the optimal policy for (1.4), and cite several results from Donchev (1995) that characterize $\{x_t\}$. (Complete proofs of all auxiliary results are given in the appendices.) In Section 4 we consider the case $\mu = 0$. If $\mu = 0$, the problem is simplified in such a way that we obtain explicit formulas for the discounted expected returns that correspond to the optimal policy. We use the formulas to calculate the maximum average return per unit time.

2. DESCRIPTION OF OPTIMAL POLICY

Before presenting the formal proof of the optimal portfolio allocation policy, we describe the results in a nontechnical way. The key to understanding the derivation of the optimal policy consists of two parts: first, that the optimal policy depends only on the process $\{x_t\}$, and second, that the process $\{x_t\}$ depends on the investor's observations on cash flows. To specify the investor's observations on cash flows, let $N_i^*(t)$, for $i = 1, 2$, be the flow of cash payments from asset i. Note that $N_i^*(t)$ is the flow of cash payments generated by asset i, and so does not depend on the investor's portfolio allocation $\{U_t\}$, while $N_i(t)$ is the flow of cash payments generated by asset i that accrue to the investor, and so does depend on the investor's portfolio allocation $\{U_t\}$.

To capture the investors observations on cash flows we specify the process

$$S(t) = \frac{1 + \int_0^t N_1^*(s)\,ds}{1 + \int_0^t N_2^*(s)\,ds}, \tag{2.1}$$

which the investor is assumed to monitor. By construction $S(0) = 1$. An advantage of monitoring the ratio is that the investor needs to keep track of only one process. Unfortunately, it is not easy to obtain solutions for such a ratio if $\mu > 0$. To treat the case for which $\mu > 0$, we also derive results when the investor must monitor two cash flow processes, defined as

$$\begin{aligned} S_1(t) &= \int_0^t N_1^*(s)\,ds, \\ S_2(t) &= \int_0^t N_2^*(s)\,ds, \end{aligned} \tag{2.2}$$

where $S_1(0) = S_2(0) = 0$. We also assume that in altering the portfolio allocation the investor bears transaction costs. We begin at period 0 and study the actions of an investor over the time interval $[0, t_A]$. Suppose that $x \in [0, 0.5]$, so that the investor believes that it is more likely that asset 1 has the higher mean return. Initially, the investor places the entire dollar in asset 1.

To relate the relative outcome of a given investment plan, $v^\pi(x)$, to the underlying cash flow processes, we split the positive time axis into sequential intervals. If the first interval represents a period in which asset 1 has the higher intensity cash flow λ, then the second interval represents a period in which asset 2 has the higher intensity cash flow. The points in time at which the mean cash flow from asset 1 switches between λ and μ are determined by a third Poisson process, which is independent of both $N_1(t)$ and $N_2(t)$, that has intensity a. The length of any interval is random and is determined by the arrivals from a Poisson process with intensity a.

To understand the investors portfolio allocation decisions, we present the decisions confronted by the investor in each case. The technical results that underpin the investors decision are presented in the next section. For each of the following cases, we first assume that the initial condition $x \in [0, 0.5)$. (The case $x \in (0.5, 1.0]$ is symmetric as we show in (2.4). The case $x = 0.5$ is different, and we describe it below.) We illustrate the optimal strategy for each case by examining one trajectory.

Case 1. We begin with the case in which the investor monitors the ratio of cash flows from the two assets given by (2.1), thus $\mu = 0$.

Suppose the process $S(t)$ stays below $S(0) + \varepsilon_0$ over the interval $[0, t_A]$. (Here, $\varepsilon_0 = \varepsilon_0(\lambda) > 0$. We allow the value of the transaction costs to depend on both time and the intensity λ.)[4] Because the information from the

[4] Because the investor is monitoring the ratio, the value $\varepsilon_0(\lambda)$ must be chosen in such a way that the value of t for which the ratio first crosses the value $S(0) + \varepsilon_0(\lambda)$ is an exponential random variable with mean $1/\lambda$.

observed cash flows is not precise enough to overcome the transactions costs, which are less than $\varepsilon_0(\lambda)$, the investor does not alter his portfolio. The process x_t increases continuously and approaches 0.5, because the value 0.5 corresponds to the belief that the assets are equally likely to have cash flows generated by a Poisson process with intensity λ. More precisely, as we derive in Theorem 3.1, x_t satisfies the following differential equation

$$x_t' = \lambda x_t(1 - x_t) + a(1 - 2x_t), \tag{2.3}$$

where $x_t' = dx_t/dt$. Recall that $a > 0$ is the rate at which the mean return of the assets switches.

The process $S(t)$ goes above the level $S(0) + \varepsilon_0(\lambda)$ for the first time at time $t_B > t_A$. At the point t_B the existing observations indicate that asset 1 is the better asset, so x_t jumps instantaneously to 0, that is $x_{t_B} = 0$. At time t_B the bound for which the investor resets x_t and so potentially alters portfolio weights becomes $S(t_B) + \varepsilon_B(\lambda)$. If the value of $S(t)$ stays below $S(t_B) + \varepsilon_B(\lambda)$ over the period $[t_B, t_C]$, then the process x_t again increases continuously and approaches 0.5 according to (2.3). At the point t_C, the control $x_{t_C} = \frac{1}{2}$[5]. At time t_C the bound for which the investor resets x_t becomes $S(t_C) + \varepsilon_C(\lambda)$. The control remains at the value $\frac{1}{2}$ as long as $S(t)$ remains within $[S(t_C) - \varepsilon_C(\lambda), S(t_C) + \varepsilon_C(\lambda)]$. If $S(t)$ goes above the level $S(t_C) + \varepsilon_C(\lambda)$ for the first time at t_{D^+}, then x_t jumps instantaneously to zero. If $S(t)$ goes below the level $S(t_C) - \varepsilon_C(\lambda)$ for the first time at t_{D^-}, then x_t jumps instantaneously to one. If $S(t)$ remains above $S(t_{D^-}) - \varepsilon_{D^-}(\lambda)$, then x_t decreases continuously and approaches 0.5. Again from Theorem 3.1, x_t satisfies the following differential equation

$$x_t' = -\lambda x_t(1 - x_t) + a(1 - 2x_t), \tag{2.4}$$

with initial condition $x_{D^-} = 1.0$.

Our optimal policy suggests that if x_t is in the interval [0, 0.5), then the investor should allocate his entire portfolio to asset 1. If x_t is in the interval (0.5, 1], then the investor should allocate his entire portfolio to asset 2. If $x_t = 0.5$, then the investor's decision is more complicated. With x_t constant at 0.5, the investor believes it equally likely that either asset is the high return asset and so does not change his investment position.

The results in Section 3 reveal that under the optimal policy, the investor allocates his portfolio to the high return asset at least half of the time, on average. If $\lambda/a \to \infty$ then the average time that the investor

[5] Because the process x_t approaches the value $\frac{1}{2}$ continuously from below, the process cannot exceed $\frac{1}{2}$ without first equalling $\frac{1}{2}$.

allocates his portfolio in the high return asset tends to 1. More precisely, if $\lambda/a \to \infty$ then for

$$\liminf_{t\to 0} E_x^\pi\left[\frac{1}{\lambda t}(N_1(t)+N_2(t))\right] = 1+g,$$

$g \to 0$. An explicit expression for $g = \lim_{\beta\downarrow 0}\beta v_\beta(x)$, with $v_\beta(x)$ is the maximizer of (1.4), is contained in (4.3). In other words, if the underlying distributions of the assets switch rarely, then the investor almost always invests in the high return asset.

Case 2. We now consider the general case in which $\mu \geq 0$, so the investor monitors the two processes given by (2.2).

Over the interval $[0, t_A]$, the process $S_1(t)$ stays below $S_1(0)+\varepsilon_0(\lambda)$ and the process $S_2(t)$ stays below $S_2(0)+\delta_0(\mu)$. (The value of ε_0 depends on λ because the initial value x indicates that the investor believes asset 1 has cash flows with intensity λ.) The process x_t increases continuously and approaches 0.5, because the value 0.5 corresponds to the belief that the assets are equally likely to have cash flows generated by a Poisson process with intensity λ. More precisely, as we derive in Theorem 3.1, x_t satisfies the following differential equation

$$x_t' = (\lambda-\mu)(2U_t-1)x_t(1-x_t) + a(1-2x_t), \qquad 0 \leq t \leq t_A$$

with initial condition $x \in [0, 0.5)$.

The process $S_1(t)$ goes above the level $S_1(0)+\varepsilon_0(\lambda)$ for the first time at time $t_B > t_A$, while the process $S_2(t)$ remains below $S_2(0)+\delta_0(\mu)$ over the interval $[t_A, t_B]$. (There is zero probability that both processes will leave their respective intervals at precisely the same moment.) At the point t_B the existing observations indicate that asset 1 is the better asset, so x_t jumps instantaneously to a lower level

$$x_{t_B} = \frac{\mu x_{t_B}^-}{\mu x_{t_B}^- + \lambda(1-x_{t_B}^-)} < x_{t_B}^-,$$

where $x_{t_B}^- = \lim_{s\uparrow t_B} x_s$.

The process $S_2(t)$ goes above the level $S_2(0)+\delta_0(\mu)$ for the first time at time t_B, while the process $S_1(t)$ remains below $S_1(0)+\varepsilon_0(\lambda)$ over the interval $[t_A, t_B]$. At the point t_B the existing observations indicate that asset 2 is the better asset, so x_t jumps instantaneously to a higher level

$$x_{t_B} = \frac{\lambda x_{t_B}^-}{\lambda x_{t_B}^- + \mu(1-x_{t_B}^-)}.$$

The value of x_{t_B}, which depends on the value $x^-_{t_B}$, falls into one of the following ranges: $x_{t_B} < \frac{1}{2}$, or $x_{t_B} > \frac{1}{2}$. If $x_{t_B} < \frac{1}{2}$, then the investor continues to invest his entire portfolio in asset 1. If, however, $x_{t_B} > \frac{1}{2}$, then the investor switches his entire portfolio in asset 2 at time t_B. Because it is possible that $\mu > 0$, there is no explicit formula for $1+g$ and the β-discounted total reward, $v_\beta(x)$. However, we construct two-sided bounds for these quantities (see Theorem 3.3).

3. FORMAL RESULTS

We begin this section with characterization of the process $\{x_t\}$. For notational ease, let $N^1_t = N_1(t)$, $N^2_t = N_2(t)$, and $\varepsilon = (\lambda - \mu)$. We assume that $\lambda > \mu \geq 0$.

Theorem 3.1. The process $\{x_t\}$ is piecewise-deterministic and its jump-rate is equal to $U_t[\mu x_t + \lambda(1-x_t)] + (1-U_t)[\lambda x_t + \mu(1-x_t)]$. The jump-times t_n, $n \geq 1$, of $\{x_t\}$ coincide with those of the process $\{N^1_t + N^2_t\}$ and its state immediately after the n-th jump for $n = 1, 2, \ldots$ is

$$x_n = \begin{cases} \dfrac{\mu x_{t_n}}{\mu x^-_{t_n} + \lambda(1 - x^-_{t_n})}, & \text{if } N^1_{t_n} - N^{1-}_{t_n} = 1 \\ \dfrac{\lambda x_{t_n}}{\lambda x^-_{t_n} + \mu(1 - x^-_{t_n})}, & \text{if } N^2_{t_n} - N^{2-}_{t_n} = 1. \end{cases} \tag{3.1}$$

In every interval $[t_n, t_{n+1})$, $n \geq 0$, $t_0 = 0$, the sample paths of $\{x_t\}$ satisfy the following first-order differential equation

$$\begin{aligned} x'_t &= \varepsilon(2U_t - 1)x_t(1-x_t) + a(1 - \lambda x_t) \\ x_{t_n} &= x_n, \end{aligned} \tag{3.2}$$

where $x_0 = x$ and x_n, $n \geq 1$ are given by formula (3.1).

Proof. See Appendix 1. □

The next theorem contains our main result. We provide the optimal policy, which is the policy that yields the value

$$v_\beta(x) = \max_\pi -E^\pi_x \int_0^\infty e^{-\beta t}[U_t x_t + (1-U_t)(1-x_t)]\, dt. \tag{3.3}$$

Theorem 3.2. There exists a constant $K \geq 0$ that depends only on μ and such that for all $a \geq K$ the policy (1.5) maximizes (1.4). Moreover, the function $K = K(\mu)$ is non-decreasing and $K(0) = 0$.

We only sketch the proof of this theorem. Making use of Theorem 3.1 we define operators L_1, L_2 and L acting on functions $v \in C^1([0,1])$ by the formulas

$$L_1 v(x) = [\varepsilon x(1-x) + a(1-2x)]v'(x) + [\mu x + \lambda(1-x)]\left[v\left(\frac{\mu x}{\mu x + \lambda(1-x)}\right) - v(x)\right] - x, \quad (3.4)$$

$$\begin{aligned} L_2 v(x) =& [-\varepsilon x(1-x) + a(1-2x)]v'(x) \\ &+ [\lambda x + \mu)(1-x)]\left[v\left(\frac{\lambda x}{\lambda x + \mu(1-x)}\right) - v(x)\right] - (1-x) \\ =& L_1 v(1-x), \end{aligned} \quad (3.5)$$

$$Lv(x,u) = uL_1 v(x) + (1-u)L_2 v(x). \quad (3.6)$$

Then, the Bellman equation corresponding to (3.3) turns into

$$\max_{u \in [0,1]} Lv(x,u) = \beta v(x). \quad (3.7)$$

Because the operator L is a linear function of u, (1.5) and (3.4) imply that in order to prove Theorem 3.2 it suffices to find a function $v \in C^1([0,1])$ that satisfies the conditions

$$L_2 v(x) = \beta v(x), \qquad x \geq \tfrac{1}{2}, \quad (3.8)$$

$$L_1 v(x) \leq \beta v(x), \qquad x \geq \tfrac{1}{2}, \quad (3.9)$$

$$v(x) = v(1-x), \qquad x \in [0,1]. \quad (3.10)$$

Such a function should also satisfy (3.7) and it should be a value function of the problem (3.3). Note that condition (3.9) implies that

$$v'(1/2) = 0. \quad (3.11)$$

Condition (3.8) is a functional-differential equation for the unknown function $v(x)$. After introducing a logarithmic scale $y = \ln(1-x) - \ln x$ and a new unknown function $V(y) = v(1/(1+e^{y}))$ the expressions for the operators L_1 and L_2 take the following form

$$L_1 V(y) = -(\varepsilon + 2\sinh(y)a)V'(y) - \mu\frac{1+e^{y+\gamma}}{1+e^{y}}[V(y) - V(y+\gamma)] - \frac{1}{1+e^{y}}, \tag{3.12}$$

$$L_2 V(y) = (\varepsilon - 2\sinh(y)a)V'(y) - \lambda\frac{1+e^{y-\gamma}}{1+e^{y}}[V(y) - V(y-\gamma)] - \frac{e^{y}}{1+e^{y}}, \tag{3.13}$$

where $\sinh(y)$ is the hyperbolic sin of y and $\gamma = \ln\lambda - \ln\mu$. So, conditions (3.8)–(3.11) turn into

$$L_2 V(y) = \beta V(y), \qquad y \leq 0, \tag{3.14}$$

$$L_1 V(y) \leq \beta V(y), \qquad y \leq 0, \tag{3.15}$$

$$V(y) = V(-y), \tag{3.16}$$

$$V'(0) = 0. \tag{3.17}$$

Let us note that in view of (3.13) the equation (3.14) has a constant delay. It is convenient to define a new unknown function $f(y)$, $y \leq 0$, by the formula

$$f(y) = (1 + e^{-y})V(y). \tag{3.18}$$

Substituting $V(y)$ from (3.18) in (3.14) and utilizing (3.13) we obtain the following equation for the function $f(y)$:

$$f'(y) = \frac{\mu[f(y) - f(y-\gamma)] + (a + \beta - a\,e^{-y})f(y) + 1}{\varepsilon - 2\sinh(y)a}. \tag{3.19}$$

As shown by Donchev (1995a, Thm 4.3 (see Appendix 3)) the general solution of this equation on the negative half-line is

$$f(y) = -\frac{1 + \beta/a + e^{-y}}{\beta(2 + \beta/a)} + K\eta(y), \qquad \mathrm{K} = \mathrm{const}, \tag{3.20}$$

where $\eta(y)$ is a non-trivial solution of the corresponding homogeneous equation, which has the value $1+\beta/a+e^{-y}+o(1)$ as $y\to-\infty$. To every solution of Eq. (3.19) there corresponds a unique solution of Eq. (3.14). Moreover, we are in a position to choose the constant K in (3.20) in such a way that (3.17) holds. Taking an even continuation of function $V(y)$ on the positive half-line we can fulfill condition (3.16) as well. To complete the proof of Theorem 3.2 it suffices to verify the inequality (3.15). The main difficulty in the proof of this inequality is that the function $\eta(y)$ in the right-hand side of formula (3.20) cannot be represented in closed form. To overcome the difficulty we obtain estimates which allow us to prove (3.14) for all small enough μ/a.

Let us set $K=\beta^{-1}(2+\beta/a)^{-1}$ in (3.20) and denote by $\xi(y)$ the corresponding solution of Eq. (3.19). Since the function $\eta(y)$ has the value $1+\beta/a+e^{-y}+o(1)$ as $y\to-\infty$, it follows that $\xi(y)=o(1)$ as $y\to-\infty$. Consider the functions

$$d(y)=\frac{e^{-y}(1+x_2e^{y})^{-b}(1+x_1e^{y})^{-c}-(e^{-y}+1+(\varepsilon+\beta)/a)}{am}\tag{3.21}$$

$$D(y)=\frac{e^{-y}(1+x_2e^{y})^{-B}(1+x_1e^{y})^{-C}-(e^{-y}+1+(\lambda+\beta)a)}{aM},\tag{3.22}$$

where

$$x_{1,2}=\frac{\varepsilon}{2a}\pm\sqrt{1+\left(\frac{\varepsilon}{2a}\right)^2}\tag{3.23}$$

are roots of the equation $x^2-(\varepsilon/a)x-1=0$ and the other constants are equal to

$$b=\frac{1+\beta/a+x_2}{x_1-x_2},\qquad c=-\frac{1+\beta/a+x_1}{x_1-x_2},\tag{3.24}$$

$$B=\frac{1+(\mu+\beta)/a+x_2}{x_1-x_2},\qquad C=-\frac{1+(\mu+\beta)/a+x_1}{x_1-x_2},\tag{3.25}$$

$$m=-(x_1-x_2)^2bc=\frac{\beta}{a}\left(2+\frac{\beta}{a}\right)+\frac{\varepsilon}{a}\left(1+\frac{\beta}{a}\right),\tag{3.26}$$

$$M=-(x_1-x_2)^2BC=\left(1+\frac{\mu+\beta}{a}\right)\left(1+\frac{\lambda+\beta}{a}\right)-1.\tag{3.27}$$

The functions $d(y)$ and $D(y)$ satisfy the ordinary differential equations

$$d'(y) = \frac{1 - (a\,e^{-y} - a - \beta)d(y)}{\varepsilon - 2\sinh(y)a}, \tag{3.28}$$

$$D'(y) = \frac{1 - (a\,e^{-y} - a - \mu - \beta)D(y)}{\varepsilon - 2\sinh(y)a}, \tag{3.29}$$

respectively, as well as the condition

$$\lim_{y \to -\infty} d(y) = \lim_{y \to -\infty} D(y) = 0. \tag{3.30}$$

The following theorem (Donchev (1995, Thm 3.1) allows us to prove an inequality that is stronger than (3.15).

Theorem 3.3. If $\mu/a > 0$ is small enough, then

$$0 < d(y) < \xi(y) < D(y), \tag{3.31}$$

$$0 < d'(y) < \xi'(y) < D'(y), \tag{3.32}$$

for all $y < 0$.

Proof. See Appendix 2.

All details of the proof of inequality (3.15) can be found in Donchev (1995a). So, all conditions (3.8)–(3.10) hold true for the function $v(x) = V(\ln(1-x)/x)$. This completes the proof of Theorem 3.2. □

The function $v(x) = V(\ln(1-x)/x)$ is the value function for (1.4). Note that according to Theorem 3.3 both $v(x)$ and its first derivative can be estimated by means of the functions $d(y)$ and $D(y)$.

4. THE CASE $\mu = 0$

If μ equals 0 then only one of the bandit's arms yields a positive gain. According to Theorem 3.2, if $\mu = 0$ then the policy (1.5) is an optimal policy for every $a > 0$. Comparing formulas (3.21) and (3.22), we obtain that in this case both functions $d(y)$ and $D(y)$ coincide. Hence, in view of Theorem 3.3, each of these functions equals $\xi(y)$. This allows us to find an explicit expression for the value function $v(x)$.

Denote by g the maximum average income per unit time defined by the formula

$$g = -\sup_{\pi} \lim_{t\to\infty} t^{-1} E_x^{\pi} \int_0^t [U_t x_t + (1-U_t)(1-x_t)]\,dt. \tag{4.1}$$

Theorem 4.1. If $\mu = 0$, then for any $a > 0$ the policy (1.5) is Blackwell optimal (that is, it is uniformly optimal with respect to β) for the problem (1.4). Moreover, the value function $v(x) = v_\beta(x)$ and the maximum average income per unit time equal

$$Hv_\beta(x)$$
$$= \begin{cases} \beta^{-1}\left[\dfrac{\lambda+2\beta}{a}(1+x_2)^c(1+x_1)^b - \dfrac{\lambda+\beta}{a}\right]\left[1+\dfrac{\beta}{a}(1-x)\right] \\ \quad + a^{-1}\left\{[x+x_1(1-x)]^{-c}[x+x_2(1-x)]^{-b} - 1 - \dfrac{\lambda+\beta}{a}(1-x)\right\}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } x \geq \tfrac{1}{2} \\ -\beta^{-1}\left[\dfrac{\lambda+2\beta}{a}(1+x_2)^c(1+x_1)^b - \dfrac{\lambda+\beta}{a}\right]\left[1+\dfrac{\beta}{a}x\right] \\ \quad + a^{-1}\left[(1-x+x_1x)^{-c}(1-x+x_2x)^{-b} - 1 - \dfrac{\lambda+\beta}{a}x\right], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } x \leq \tfrac{1}{2} \end{cases} \tag{4.2}$$

$$g = -\frac{(1+x_2)^{-(1+x_1)/(x_1-x_2)}(1+x_1)^{(1+x_2)/(x_1-x_2)} - 1}{2(1+x_2)^{-(1+x_1)/(x_1-x_2)}(1+x_1)^{(1+x_2)/(x_1-x_2)} - 1}, \tag{4.3}$$

respectively, where the constant H in (4.2) is

$$H = \frac{\lambda}{a}\left(1+\frac{\beta}{a}\right) + \left(2+\frac{\beta}{a}\right)\left[(1+x_2)^c(1+x_1)^b\frac{\lambda+2\beta}{a} - \frac{\lambda}{a}\right], \tag{4.4}$$

and b and c are given by the expressions (3.24) and

$$x_{1,2} = \frac{\lambda}{2a} \pm \sqrt{1+\left(\frac{\lambda}{2a}\right)^2}.$$

Proof. It follows from the paragraph preceding Theorem 4.1 and the definition of $\xi(y)$ that

$$d(y) = \frac{\eta(y) - 1 - \beta/a - e^{-y}}{\beta(2 + \beta/a)}, \qquad y \le 0 \tag{4.5}$$

$d(y)$ being defined by (3.21). Hence, we get $\eta(y) = \beta(2 + \beta/a)\, d(y) + 1 + \beta/a + e^{-y}$. Substituting this expression in (3.20) we obtain

$$\begin{aligned} f(y) = &\left[K - \beta^{-1}\left(2 + \frac{\beta}{a}\right)^{-1}\right]\left(1 + \frac{\beta}{a} + e^{-y}\right) \\ &+ K\beta(2 + \beta/a)\, d(y), \qquad y \le 0 \end{aligned} \tag{4.6}$$

and in view of (3.18)

$$\begin{aligned} V(y) = &\left[K - \beta^{-1}\left(2 + \frac{\beta}{a}\right)^{-1}\right]\left(1 + \frac{\beta}{a}\cdot\frac{1}{1 + e^{-y}}\right) \\ &+ K\beta\left(2 + \frac{\beta}{a}\right)\frac{d(y)}{1 + e^{-y}}, \qquad y \le 0. \end{aligned} \tag{4.7}$$

On the other hand, formula (3.21) implies

$$\begin{aligned} \frac{am}{1 + e^{-y}} d(y) = &(1 + x_2 e^{y})^{-b}(1 + x_1 e^{y})^{-c}\frac{1}{1 + e^{y}} \\ &- \left(1 + \frac{\lambda + \beta}{a}\frac{1}{1 + e^{-y}}\right), \qquad y \le 0. \end{aligned}$$

Hence, the expression (4.7) turns into

$$\begin{aligned} V(y) = &\left[K - \beta^{-1}\left(2 + \frac{\beta}{a}\right)^{-1}\right]\left(1 + \frac{\beta}{a}\frac{1}{1 + e^{-y}}\right) + \frac{K}{m}\frac{\beta}{a}\left(2 + \frac{\beta}{a}\right) \\ &\times \left[(1 + x_2 e^{y})^{-b}(1 + x_1 e^{y})^{-c}\frac{1}{1 + e^{y}} \right. \\ &\left. - \left(1 + \frac{\lambda + \beta}{a}\frac{1}{1 + e^{-y}}\right)\right], \qquad y \le 0. \end{aligned}$$

Because $b + c = -1$, we rewrite $v_\beta(x) = V(\ln(1-x)/x)$ in the following form

$$v_\beta(x) = \left[K - \beta^{-1}\left(2 + \frac{\beta}{a}\right)^{-1}\right]\left[1 + \frac{\beta}{a}(1-x)\right] + \frac{K\beta}{m\,a}\left(2 + \frac{\beta}{a}\right) \times \Big\{[x + x_2(1-x)]^{-b}\,[x + x_1(1-x)]^{-c} - 1 - \frac{\lambda+\beta}{a}(1-x)\Big\}, \qquad x \geq 1/2. \tag{4.8}$$

We determine the unknown constant K from the condition (3.11). The derivative of $v_\beta(x)$ is

$$v_\beta'(x) = -\frac{\beta}{a}\left[K - \beta^{-1}\left(2 + \frac{\beta}{a}\right)^{-1}\right] + \frac{K\beta}{m\,a}\left(2 + \frac{\beta}{a}\right) \times \left\{\frac{\lambda+\beta}{a} - \left(\frac{\beta}{\alpha} + \frac{\lambda}{\alpha}x\right)[x + x_1(1-x)]^{b}\,[x + x_2(1-x)]^{c}\right\}, \qquad x \geq 1/2.$$

Thus, condition (3.11) implies that

$$K = \beta^{-1}\left(2 + \frac{\beta}{a}\right)^{-1}\frac{m}{H}, \tag{4.9}$$

H being given by (4.4).

Substituting (4.9) in (4.8) and taking into account condition (3.10) we get (4.2). The Laurent expansion of $v_\beta(x)$ implies that

$$g = \lim_{\beta \downarrow 0} \beta v_\beta(x).$$

From the above limiting expression for g and the equalities (4.2) and (4.4) we obtain (4.3). □

Comparing formulas (1.1), (4.1) and (1.4) we see that $1 + g$ is equal to the average time the investor makes the right decision following the policy (1.5). It is easy to verify that $1 + g$ is between $\frac{1}{2}$ and 1. In fact, for any fixed $\lambda > 0$

$$\lim_{a \to 0}(1+g) = 1, \qquad \lim_{a \to \infty}(1+g) = \frac{1}{2}.$$

The sense of these formulas is quite apparent. If the underlying intensity switches rarely, then the investor is able to determine the high intensity asset almost all the time. Conversely, if the underlying intensity switches too often, then the investor is unable to improve on a policy based on random coin flipping.

5. PROOF OF THEOREM 3.1 (APPENDIX 1)

The theorem reduces the Poisson two-armed bandit problem, which is a problem with incomplete information, to an equivalent problem for a completely observed control process. Rigorous proof of the theorem involves lengthy definitions of arbitrary history dependent policies. Proof of the theorem can be accomplished with the martingale methods that are used if there is no switching, Presman and Sonin (1990, Sections 1.7, 4.1, and 4.2), which in our model corresponds to the case $a = 0$. Here we only sketch the proof utilizing Bayes' rule to reevaluate $x_{t+\Delta t}$ for given x_t, U_t and observations on the interval $[t, t+\Delta t]$. Following Bertsekas and Shreve (1978) and Dynkin and Yushkevich (1979) we define a model $M = \{S, C, Z, k, r\}$ that consists of the following elements:

(1) The state space $S = \{1, 2\}$ consists of two elements. The state $s = i, i = 1, 2$, means that the hypothesis H_i is true;

(2) The space of all admissible controls is $C = [0, 1]$. This is the state space of the process $\{U_t\}$;

(3) The observation space $Z = \{0, 1, 2\}$. We observe the state $z = 0$ if the process $\{N_t^1 + N_t^2\}$ has no jumps over the interval $(t, t+\Delta t)$. If the process $\{N_t^1\}$ ($\{N_t^2\}$) has jumps over the interval $(t, t+\Delta t)$, then we observe the state $z = 1$ ($z = 2$).

(4) The transition kernel $k_{\Delta t}(\cdot\,|s)$ is a measure on S depending on $s \in S$ which we define as follows

$$k_{\Delta t}(1|1) = k_{\Delta t}(2|2) = 1 - a\Delta t,$$

$$k_{\Delta t}(1|2) = k_{\Delta t}(2|1) = a\Delta t;$$

(5) The observation kernel $r_{\Delta t}(\cdot\,|s, u)$ is a conditional probability on Z depending on $(s, u) \in S \times C$. We set

$$r_{\Delta t}(1|1, u) = \mu u\Delta t, r_{\Delta t}(2|1, u) = \lambda(1-u)\Delta t,$$

$$r_{\Delta t}(1|2, u) = \lambda u\Delta t, r_{\Delta t}(2|2, u) = \mu(1-u)\Delta t,$$

$$r_{\Delta t}(0|1, u) = 1 - [\mu u + \lambda(1-u)]\Delta t = 1 - (\lambda - \varepsilon u)\Delta t,$$

$$r_{\Delta t}(0|2, u) = 1 - [\lambda u + \mu(1-u)]\Delta t = 1 - (\mu + \varepsilon u)\Delta t,$$

where in the formulas for k and r the lower index denotes the length of the time interval.

We apply the well known filtration algorithm to the model M. The algorithm is used to calculate the *a priori* and *a posteriori* probabilities of the states of space S in sequential time intervals. It follows from the definition of the process $x_t = P_t^\pi\{H_1|F_t\}$, that its value at time t is equal to the *a posteriori* probability of the state $1 \in S$. Denote by p_t the *a priori* probability of this state and assume that in the interval $(s, t+\Delta t)$, $s < t$, $\Delta t > 0$, the control U_t is used. Since we shall pass to a limit as $s \uparrow t$ and $\Delta t \downarrow 0$ the assumption is not restrictive. Then, the filtration equations take the following form

$$\begin{aligned} p_{t+\Delta t} &= k_{\Delta t+t-s}(1|1)x_s + k_{\Delta t+t-s}(1|2)(1-x_s) \\ &= [1-a(\Delta t + t - s)]x_s + a(\Delta t + t - s)(1-x_s), \end{aligned} \tag{5.1}$$

$$x_{t+\Delta t}(z) = \frac{r_{\Delta t+t-s}(z|1, U_t)p_{t+\Delta t}}{r_{\Delta t+t-s}(z|1, U_t)p_{t+\Delta t} + r_{\Delta t+t-s}(z|2, U_t)(1-p_{t+\Delta t})}, \qquad z \in Z. \tag{5.2}$$

If $t = t_n$ is a jump-time of the process $\{N_t^i\}$, $i = 1, 2$, then we observe jumps of the process in every interval $(s, t+\Delta t)$, $s < t$, $\Delta t > 0$. Setting $z = i$, $i = 1, 2$, in (5.2), from (5.1), the definitions of $r_{\Delta t+t-s}(i|1, U_t)$ and $r_{\Delta t+t-s}(i|2, U_t)$, and passing to a limit first as $s \uparrow t$ and after that as $\Delta t \downarrow 0$ we get (3.1). If t is not a jump-time of the process $\{N_t^1 + N_t^2\}$ then taking Δt small enough and s sufficiently close to t, then no jumps of the process will be observed in $(s, t+\Delta t)$. Therefore, in the case $z = 0$, Eq. (5.2) becomes

$$x_{t+\Delta t} = \frac{[1-(\lambda-\varepsilon u)(\Delta t + t - s)]p_{t+\Delta t}}{[1-(\lambda-\varepsilon u)(\Delta t+t-s)]p_{t+\Delta t} + [1-(\mu+\varepsilon u)(\Delta t+t-s)](1-p_{t+\Delta t})}.$$

Substituting $p_{t+\Delta t}$ from (5.1) in the last formula, multiplying by the denominator of the expression in the right-hand side and passing to a limit as $s \uparrow t$ we get

$$\begin{aligned} x_{t+\Delta t} - x_t^- = {}&\Delta t[\lambda(x_{t+\Delta t} - x_t^-) - \varepsilon(2U_t - 1)x_{t+\Delta t}x_t^- \\ &- \varepsilon(1-U_t)x_{t+\Delta t} + \varepsilon U_t x_t^- + a(1-2x_t^-)] + o(\Delta t). \end{aligned}$$

The limit as $\Delta t \downarrow 0$ of the expression in the brackets exists and equals

$$\lambda(x_t - x_t^-) - \varepsilon(2U_t - 1)x_t x_t^- - \varepsilon(1-U_t)x_t + \varepsilon U_t x_t^- + a(1-2x_t^-). \tag{5.3}$$

Thus $\lim_{\Delta t \downarrow 0} (x_{t+\Delta t} - x_t^-)/\Delta t$ exists. Hence, we obtain that $x_t = x_t^-$. From this fact and (5.3) we deduce that the right derivative of x_t at point t exists and equals the expression in the right-hand side of formula (3.2). Repeating the same considerations for the interval (s, t) we see that the left derivative of x_t exists and that the two derivatives coincide. Finally, if $\Delta t \downarrow 0$ then the probability that N_t^1 (N_t^2) jumps in the interval $(t, t+\Delta t)$ is equal to

$$U_t[\mu x_t + \lambda(1-ax_t)]\Delta t + o(\Delta t)((1-U_t)[\lambda x_t + \mu(1-x_t)]\Delta t + o(\Delta t)).$$

Therefore, $U_t[\mu x_t + \lambda(1-ax_t)] + (1-U_t)[\lambda x_t + \mu(1-x_t)]$ is the jump rate of both $\{x_t\}$ and $\{N_t^1 + N_t^2\}$. $\square$

6. PROOF OF THEOREM 3.3. (APPENDIX 2)

We precede the proof of this theorem with four lemmas.

Lemma 6.1. The following inequalities hold

$$(k + x_2 u)^{-b}(k + x_1 u)^{-c} > k + \left(1 + \frac{\varepsilon + \beta}{a}\right)u, \tag{6.1}$$

$$(k + x_2 u)^{b+1}(k + x_1 u)^{c+1} > k - \left(1 + \frac{\beta}{a}\right)u, \tag{6.2}$$

$$(k + x_2 u)^{-B}(k + x_1 u)^{-C} > k + \left(1 + \frac{\lambda + \beta}{a}\right)u, \tag{6.3}$$

$$(k + x_2 u)^{B+1}(k + x_1 u)^{C+1} > k - \left(1 + \frac{\mu + \beta}{a}\right)u, \tag{6.4}$$

where $0 < u < x_1 k$, $k > 0$, and x_1, x_2, b, c, B, C, being defined by (3.23)–(3.27).

Proof. Consider the functions

$$G_1(u) = (k + x_2 u)^{-B}(k + x_1)^{-C} - k - \left(1 + \frac{\lambda + \beta}{a}\right)u,$$

$$G_2(u) = (k + x_2 u)^{B+1}(k + x_1)^{C+1} - k + \left(1 + \frac{\mu + \beta}{a}\right)u.$$

Calculations show that

$$Bx_2 + Cx_1 = -1 - \frac{\lambda + \beta}{a} \tag{6.5}$$

$$bx_2 + cx_1 = -1 - \frac{\varepsilon + \beta}{a}. \tag{6.6}$$

From the identities $x_1 x_2 = -1$, $x_1 + x_2 = \varepsilon/a$, as well as (6.5) we get

$$G_1'(u) = (k + x_2 u)^{-B-1}(k + x_1 u)^{-C-1}\left[k\left(1 + \frac{\lambda+\beta}{a}\right) - u\right] - 1 - \frac{\lambda+\beta}{a},$$

$$G_1''(u) = Mk^2(k + x_2 u)^{-B-2}(k + x_1 u)^{-C-2},$$

$$G_2'(u) = -(k + x_2 u)^{B}(k + x_1 u)^{C}\left[k\left(1 + \frac{\mu+\beta}{a}\right) + u\right] + 1 + \frac{\mu+\beta}{a},$$

$$G_2''(u) = Mk^2(k + x_2 u)^{B-1}(k + x_1 u)^{C-1}.$$

Since $B + C = -1$ it is easy to see that $G_1(0) = G_1'(0) = G_2(0) = G_2'(0) = 0$, whereas both $G_1''(u)$ and $G_2''(u)$ are positive. This implies that $G_1(u) > 0$, $G_2(u) > 0$, which proves both (6.3) and (6.4). Repeating the same argument for the functions

$$g_1(u) = (k + x_2 u)^{-b}(k + x_1 u)^{-c} - k - \left(1 - \frac{\varepsilon+\beta}{a}\right)u,$$

$$g_2(u) = (k + x_2 u)^{b+1}(k + x_1 u)^{c+1} + \left(1 + \frac{\beta}{a}\right)u - k,$$

we get (6.1) and (6.2). □

Lemma 6.2. Both functions $d(y)$ and $D(y)$ are increasing and convex provided $y < \ln x_1$.

Proof. Since the proof is similar for the two functions we shall prove the claim of the lemma only for the function $d(y)$. Calculating the first derivative of $d(y)$ we obtain

$$am\, e^{y} d'(y) = (1 + x_2\, e^{y})^{-b-1}(1 + x_1\, e^{y})^{-c-1}\left[\left(1 + \frac{\beta}{a}\right)e^{y} - 1\right] + 1.$$

Making the substitution $u = e^{y}$ and applying (6.2) we get $d'(y) > 0$, which proves the first claim of the lemma. Since $d(y)$ satisfies condition (3.29) we deduce that $d(y) > 0$. Moreover, $d(y)$ is a solution of Eq. (3.27). Hence, its second derivative is equal to

$$d''(y) = \left(\frac{1 - (a e^{-y} - a - \beta) d(y)}{\varepsilon - 2a \sinh y}\right)'$$

$$= \frac{a e^{-y} d(y) + (a e^{y} + a + \beta) d'(y)}{\varepsilon - 2a \sinh y} > 0. \tag{6.7}$$

Therefore, the function $d(Y)$ is convex. □

The next lemma plays a central role in the proof of Theorem 3.1.

Lemma 6.3. The functions $d(y)$ and $D(y)$ satisfy the inequalities

$$(a e^{-y} - \mu - a - \beta) d(y) + \mu d(y - \gamma) > (a e^{-y} - \mu - a - \beta) D(y), \tag{6.8}$$

$$(a e^{-y} - \mu - a - \beta) D(y) + \mu D(y - \gamma) < (a e^{-y} - a - \beta) d(y), \tag{6.9}$$

for all $y < \ln x_1$.

Proof. The inequality (6.8) can be rewritten in the form

$$(a e^{-y} - \mu - a - \beta)[D(y) - d(y)] < \mu d(y - \gamma). \tag{6.10}$$

Making elementary calculations we get that the left-hand side of the last inequality is equal to

$$\begin{aligned} & e^{-y}\left(e^{-y} - 1 - \frac{\mu + \beta}{a}\right) \\ & \quad \times [M^{-1} R(e^{y}, -B, -C) - m^{-1} R(e^{y}, -b, -c)] \\ & \quad + (m^{-1} - M^{-1}) e^{-2y} R(e^{y}, 1, 1) - \frac{\mu}{am}\left(e^{-y} + 1 + \frac{\varepsilon + \beta}{a}\right), \end{aligned} \tag{6.11}$$

where $R(u, \alpha, \beta) = (1 + x_2 u)^{\alpha}(1 + x_1 u)^{\beta}$.

Similarly, we can transform the right-hand side of (6.8) into

$$\frac{\mu}{am} e^{-y}(\lambda/\mu + x_2 e^{y})^{-b}(\lambda/\mu + x_1 e^{y})^{-c} - \frac{\lambda}{am} e^{-y} - \frac{\mu}{am}\left(1 + \frac{\varepsilon + \beta}{a}\right). \tag{6.12}$$

From (6.11), (6.12), the identities $b+c=B+C=-1$, and the easily verified equality

$$e^{-y}R(e^{y},1,1)=(1+x_1 e^{y})\left[e^{-y}+(x_1-x_2)B-1-\frac{\mu+\beta}{a}\right],$$

we see that (6.10) is equivalent to

$$\begin{aligned}&\left(e^{-y}-1-\frac{\mu+\beta}{a}\right)(1+x_1 e^{y})\ \frac{m[S(e^{y})]^{B}-M[S(e^{y})]^{b}+M-m}{Mm}\\&\quad+B\frac{(x_1-x_2)(M-m)}{Mm}(1+x_1 e^{y})+\frac{\varepsilon}{am}\\&\quad<\frac{\mu}{am}\left(\frac{\lambda}{\mu}+x_2 e^{y}\right)^{-b}\left(\frac{\lambda}{\mu}+x_1 e^{y}\right)^{-c},\end{aligned}\tag{6.13}$$

where $S(u)=R(u,-1,1)=(1+x_1u)/(1+x_2u)$.

It is convenient to introduce new variables $u\in[0,\ln x_1)$ and $v\in[0,\infty)$ defined by

$$u=e^{y},\ v=\ln S(e^{y})=\ln S(u).$$

Then

$$\begin{aligned}S(e^{y})&=e^{v},\ e^{-y}-1-\frac{\mu+\beta}{a}=-(x_1-x_2)\frac{C+Be^{v}}{e^{v}-1},\\1+x_1e^{y}&=(x_1-x_2)\frac{e^{v}}{x_1-x_2e^{v}}.\end{aligned}$$

Let

$$T(v)=\left(\frac{e^{Bv}}{B}-\frac{C}{c}\frac{e^{bv}}{b}+\frac{e^{Cv}}{C}-\frac{B}{b}\frac{e^{cv}}{c}+\frac{1}{BC}-\frac{1}{bc}\right).$$

We rewrite the left-hand side of (6.13) in terms of v as

$$\frac{e^{v}}{(e^{v}-1)(x_1-x_2e^{v})}T(v)+\frac{\varepsilon}{am}.$$

If we set $u=e^{y}$ in the right-hand side of (6.13) we get

$$\frac{\mu}{am}\left(\frac{\lambda}{\mu}+x_2u\right)^{-b}\left(\frac{\lambda}{\mu}+x_1u\right)^{-c}.$$

From (6.1) with $k = \lambda/\mu$, we bound the above expression from below by

$$\frac{\mu}{am}\left[1 + \left(1 + \frac{\varepsilon + \beta}{a}\right)u\right] + \frac{\varepsilon}{am}.$$

To verify (6.8) it is sufficient to prove the inequality

$$\frac{\mu}{am}\left[1 + \left(1 + \frac{\varepsilon + \beta}{a}\right)u\right] > \frac{e^{v}}{(e^{v} - 1)(x_1 - x_2 e^{v})} T(v)$$

$$= \frac{(1 + x_1 u)(1 + x_2 u)}{(x_1 - x_2)^2 u} T(v),$$

or equivalently

$$\frac{\mu}{am} \cdot \frac{u[1 + (1 + \varepsilon/a + \beta/a)u]}{(1 + x_1 u)(1 + x_2 u)} > (x_1 - x_2)^{-2} T(v). \tag{6.14}$$

The derivatives of the left-hand side and right-hand side with respect to u are equal to

$$\frac{\mu}{am}\left\{\frac{u(1 + \varepsilon/a + \beta/a)}{(1 + x_1 u)(1 + x_2 u)} + \frac{[1 + (1 + \varepsilon/a + \beta/a)u](u^2 + 1)}{(1 + x_1 u)^2 (1 + x_2 u)^2}\right\}$$

and

$$\frac{e^{v} - 1}{(x_1 - x_2)^2}\left(\frac{B}{b} e^{bv} - e^{Bv}\right) v'(u) = \left(\frac{B}{b} e^{bv} - e^{Bv}\right) \frac{u}{(1 + x_1 u)(1 + x_2 u)^2},$$

respectively. Since, if $u = 0$, both sides of (6.14) are equal to zero, then it is enough to verify that the derivative of the left-hand side is greater than the derivative of the right-hand side. Multiplying both derivatives by $u^{-1}(1 + x_1 u)(1 + x_2 u)^2$ we obtain the inequality

$$\frac{B}{b} e^{bv} - e^{Bv} < \frac{\mu}{am}\left\{\left(1 + \frac{\varepsilon + \beta}{a}\right)(1 + x_2 u) + \left[1 + \left(1 + \frac{\varepsilon + \beta}{a}\right)u\right]\frac{u + u^{-1}}{1 + x_1 u}\right\}$$

$$= \frac{\mu}{am} \cdot \frac{1 + 2(1 + \varepsilon/a + \beta/a)u + [1 + (\varepsilon/a)(1 + \varepsilon/a + \beta/a)]u^2}{u(1 + x_1 u)}.$$

The left-hand side of the last inequality is a decreasing function of both v and u. The derivative of the right-hand side equals

$$-\frac{\mu}{am}\cdot\frac{[(x_1-x_2)(1+\varepsilon/a+\beta/a)-1]u^2+2x_1u+1}{u^2(1+x_1u)}.$$

Calculating the discriminant of the numerator of the last expression we obtain $1-x_1+x_2+x_2^2<0$. Therefore, the right-hand side of the last inequality is also a decreasing function of u, which reaches a minimum value of $B/b-1$ at the point $u=x_1$. However, the value $B/b-1$ also equals the maximum value of the right-hand side reached for $v=0$, that is $x_1u=x_2u$. This completes the proof of inequality (6.8). The proof of inequality (6.9) is similar and we omit it. □

Corollary 6.4. The function $D(y)-d(y)$, $y<0$, is increasing.

Proof. It follows from inequality (6.8) and Lemma 6.2 that

$$\begin{aligned}(ae^{-y}-a-\beta)d(y) &> (ae^{-y}-\mu-a-\beta)d(y)+\mu d(y-\gamma)\\ &> (ae^{-y}-\mu-a-\beta)D(y).\end{aligned}$$

Since $d(y)$ and $D(y)$ are solutions of (3.27) and (3.28), respectively, the last inequality implies that $D'(y)>d'(y)$. In particular, in view of condition (3.29) we have $D(y)>d(y)$ for all $y<\ln x_1$. □

Lemma 6.5. If μ/a is small enough, then there exists a constant K such that $d(y)\le\xi(y)\le D(y)$ for all $y\le K$.

Proof. Lemma 6.2 implies that $d(y)-d(y-\gamma)>0$ for all $y<\ln x_1$. Because $d(y)$ satisfies (3.27) we obtain the following inequality

$$d'(y)<\frac{1-(ae^{-y}-a-\beta)d(y)+\mu[d(y)-d(y-\gamma)]}{\varepsilon-2a\sinh y}. \tag{6.15}$$

Similarly, because the function $D(y)$ is positive and satisfies (3.28) we get

$$D'(y)>\frac{1-(ae^{-y}-a-\beta)D(y)+\mu[D(y)-D(y-\gamma)]}{\varepsilon-2a\sinh y}. \tag{6.16}$$

On the other hand, the derivative $\xi'(y)$ satisfies the equation

$$\xi'(y)=\frac{1-(ae^{-y}-a-\beta)\xi(y)+\mu[\xi(y)-\xi(y-\gamma)]}{\varepsilon-2a\sinh y}. \tag{6.17}$$

We set $\zeta_1(y) = \xi(y) - D(y)$, $\zeta_2(y) = d(y) - \xi(y)$. It follows from (6.15), (6.16) and (6.17) that

$$\zeta'_1(y) < \frac{\mu[\zeta_1(y) - \zeta_1(y-\gamma)] + (a+\beta - a\,e^{-y})\zeta_1(y)}{\varepsilon - 2a\sinh y}, \tag{6.18}$$

$$\zeta'_2(y) < \frac{\mu[\zeta_2(y) - \zeta_2(y-\gamma)] + (a+\beta - a\,e^{-y})\zeta_2(y)}{\varepsilon - 2a\sinh y}. \tag{6.19}$$

Moreover, since both $d(y)$ and $D(y)$ as well as $\xi(y)$ tend to zero as $y \to -\infty$ it follows that

$$\lim_{y\to-\infty} \zeta_1(y) = \lim_{y\to-\infty} \zeta_2(y) = 0. \tag{6.20}$$

We need to prove that there exists a constant K such that $\zeta_1(y) \le 0$ and $\zeta_2(y) \le 0$ for all $y \le K$. Assume the contrary. Then, without loss of generality we may suppose that for every K there exists $y \le K$ such that $\zeta_1(y) > 0$. Let us choose the number y_1 so that the following conditions hold

$$y_1 < -\ln\left(1 + \frac{\mu+\beta}{a}\right), \tag{6.21}$$

$$\int_{-\infty}^{y_1} \frac{dt}{\varepsilon - 2a\sinh(t)} < \frac{1}{2\mu}, \tag{6.22}$$

$$\zeta_1(y_1) = \delta > 0. \tag{6.23}$$

Both functions $d(y)$ and $D(y)$ coincide provided $\mu/a = 0$. Moreover, the function $D(y) - d(y)$ is continuous with respect to μ/a and because it is positive, it follows that it increases in μ/a provided μ/a is small enough. Combining these facts with Corollary 3.1 we obtain that

$$\sup_{y<y_1} [D(y) - d(y)] < \delta, \tag{6.24}$$

for all sufficiently small μ/a.

Let z_1 be the largest number less than y_1 and such that $\zeta_1(z_1) = 0$. If such a number does not exist we set $z_1 = \infty$. Then, in view of (6.23), the function $\zeta_1(y)$ will be positive in the interval (z_1, y_1). Integrating (6.18) from z_1 to y_1 and taking into account (6.21) and (6.23) we get

$$
\begin{aligned}
\delta &= \zeta_1(y_1) \\
&< -\mu \int_{z_1}^{y_1} \frac{\zeta_1(t-\gamma)}{\varepsilon - 2a\sinh(t)}\, dt + \int_{z_1}^{y_1} \frac{\mu + a + \beta - a\,e^{-t}}{\varepsilon - 2a\sinh(t)}\, \zeta_1(t)\, dt \\
&< -\mu \int_{z_1}^{y_1} \frac{\zeta_1(t-\gamma)}{\varepsilon - 2a\sinh(t)}\, dt \\
&= -\mu \int_{z_1-\gamma}^{y_1-\gamma} \frac{\zeta_1(t)}{\varepsilon - 2a\sinh(t+\gamma)}\, dt.
\end{aligned}
$$

The last inequality and (6.22) imply that there exists a number $z_2 \in [z_1 - \gamma, y_1 - \gamma]$ such that $\zeta_1(y_2) < -2\delta$. Therefore, utilizing (6.24) and the identity $D(y_2) - d(y_2) = -\zeta_1(y_2) - \zeta_2(y_2)$ we obtain that $\zeta_2(y_2) > \delta$. Obviously, conditions (6.21) and (6.22) hold for y_2 as well. Now, we are in a position to apply to y_2 and ζ_2 the same considerations as those for y_1 and ζ_1. Denoting by z_2 the largest number $z_2 < y_2$ such that $\zeta_2(z_2) = 0$, integrating (6.19) from z_2 to y_2 and utilizing (6.21) we get

$$
\delta < \zeta_2(y_2) < -\mu \int_{z_2-\gamma}^{y_2-\gamma} \frac{\zeta_2(t)}{\varepsilon - 2a\sinh(t+\gamma)}\, dt.
$$

Thus, it follows from (6.22) that there exists $y_3 \in [z_2 - \gamma, y_2 - \gamma]$ such that $\zeta_2(y_3) < -2\delta$, whereas in view of (6.24) and the identity $D(y_3) - d(y_3) = -\zeta_1(y_3) - \zeta_2(y_3)$ we conclude that $\zeta_1(y_3) > \delta$. Repeating this procedure, we can construct a sequence $\{y_{2u+1}\}$, $u \geq 0$, such that $\lim_{u\to\infty} y_{2u+1} = -\infty$, and at the same time $\zeta_1(y_{2u+1}) > \delta$ for all $u \geq 0$. However, this contradicts (6.20). $\square$

Proof of Theorem 3.3. It follows from Lemma 6.2 and condition (3.30) that $d(y) > 0$ and $d'(y) > 0$. Let K be a constant for which the claim of Lemma 6.5 holds and σ be an arbitrary number such that $\sigma < K$. Without loss of generality we assume that $\sigma < -\ln(1 + \mu/a + \beta/a)$. Then, $a\,e^{-y} - a - \mu - \beta > 0$ and Lemmas 6.3 and 6.5 imply that

$$
\begin{aligned}
(a\,e^{-y} - a - \mu - \beta)D(y) &< (a\,e^{-y} - a - \mu - \beta)d(y) + \mu d(y-\gamma) \\
&\leq (a\,e^{-y} - a - \mu - \beta)\xi(y) + \mu\xi(y-\gamma), \qquad (6.25)
\end{aligned}
$$

$$
\begin{aligned}
(a\,e^{-y} - a - \beta)d(y) &> (a\,e^{-y} - a - \mu - \beta)D(y) + \mu D(y-\gamma) \\
&\geq (a\,e^{-y} - a - \mu - \beta)\xi(y) + \mu\xi(y-\gamma). \qquad (6.26)
\end{aligned}
$$

Since the functions $d(y)$, $D(y)$ and $\xi(y)$ satisfy Eqs. (3.27), (3.28) and (6.17), respectively, it follows from (6.25) and (6.26) that inequality (3.31) holds provided $y \leq \sigma$. Hence, the inequality (3.31) is fulfilled for some interval $(\sigma, \sigma + \sigma_0)$. Making use of (6.25) and (6.26) we get the inequality (3.32) in this interval as well. Repeating these considerations we can prove the claim of the theorem in the interval $[\sigma, -\ln(1 + \mu/a + \beta/a)]$. If $y > -\ln(1 + \mu/\alpha + \beta/\alpha)$, then $a e^{-y} - \mu - a - \beta < 0$ and both (6.25) and (6.26) cannot be used to prove the inequalities (3.31) and (3.32). However, since the last inequalities hold at the point $y = -\ln(1 + \mu/a + \beta/a)$ it follows from the continuity of the functions $d(y)$, $D(y)$ and $\xi(y)$ that they will be fulfilled also up to the point $y = -\ln(1 + \beta/a)$ provided μ/a is small enough. On the other hand, if $y \geq -\ln(1 + \beta/a)$ then we use the following inequalities in place of (6.25) and (6.26)

$$(a e^{-y} - \mu - a - \beta)D(y) < (a e^{-y} - \mu - a - \beta)\xi(y) + \mu\xi(y - \gamma)$$

$$(a e^{-y} - a - \beta)d(y) > (a e^{-y} - a - \beta)\xi(y) + \mu[\xi(y) - \xi(y - \gamma)].$$

These inequalities follow from (3.31) and (3.32) and they allow us to apply the same method in the interval $(-\ln(1 + \beta/a), 0)$ as for the interval $[\sigma, -\ln(1 + \mu/a + \beta/a)]$. Finally, since σ is chosen arbitrarily, it follows that the inequality (3.31) is satisfied on the whole negative half-time. □

7. SOLUTION OF EQUATION (3.18) (APPENDIX 3)

We consider the following system of functional-differential equations. If $t \geq 0$, then

$$f_1'(t) = (\varepsilon - 2a \sinh t)^{-1} \times \{\lambda[f_1(t) - f_1(t - \gamma)] + (a + \beta - a e^{t}) f_1(t) + 1\}, \tag{7.1}$$

if $t \leq 0$, then

$$f_2'(t) = (\varepsilon - 2a \sinh t)^{-1} \times \{\mu[f_2(t) - f_2(t - \gamma)] + (a + \beta - a e^{-t}) f_2(t) + 1\}, \tag{7.2}$$

and

$$f_1(t) = e^{t} f_2(t), \in [-\gamma, 0], \tag{7.3}$$

where $\mu, \lambda, \varepsilon, \beta, \gamma, a$ are positive constants such that

$$\lambda > \mu, \qquad \varepsilon = \lambda - \mu, \qquad \gamma = \ln \lambda - \ln \mu. \tag{7.4}$$

Equation (7.2) is in fact Eq. (3.19) whose solution is (3.20) as we shall show later in the Appendix (see Theorem 7.7). The system (7.1)–(7.3) appears in the form of a Bellman equation for the value function of the problem of optimal detection of the jump times of a Poisson process, see Donchev (1995b). It follows by dynamic programming reasoning that the system (7.1)–(7.3) has a unique solution that coincides with the value function of the optimal detection problem. From the theory of functional-differential equations, existence and uniqueness results for (7.1)–(7.3) are difficult to obtain. In particular, if we try to solve (7.1)-(7.3) there are at least two difficulties to overcome. The first is the singularity that (7.1) has at the point $t = \ln(\varepsilon/2a + \sqrt{1 + (\varepsilon/2a)^2})$ and the second is the delaying argument of (7.2) on the negative half-line.

We first examine (7.2). To do so, we consider the corresponding homogeneous equation

$$\eta'(t) = (\varepsilon - 2a \sinh t)^{-1} \times \{\mu[\eta(t) - \eta(t-\gamma)] + (a + \beta - a e^{-t})\eta(t)\}, \qquad t \le 0. \tag{7.5}$$

We shall show that Eq. (7.5) has a non-trivial solution on the negative half-line. The next theorem contains preliminary information about the asymptotic behavior of the solution.

Theorem 7.1. (i) Every solution of (7.5) satisfies

$$\eta(t) = C(1 + \beta/a + e^{-t}) + o(1), \tag{7.6}$$

as $t \to -\infty$, where C is a constant. (ii) If Eq. (7.2) has a solution then the solution has the same asymptotic behavior as (7.6).

Proof. We consider the equation

$$[-\varepsilon x(1-x) + a(1-2x)]v'(x) + [\lambda x + \mu(1-x)] \times \left[v\left(\frac{\lambda x}{\lambda x + \mu(1-x)}\right) - v(x)\right] - (1-x) = \beta v(x), \qquad x \in [1/2, 1]. \tag{7.7}$$

As shown by Donchev and Yushkevich (1997), if we introduce a logarithmic scale $t = \ln(1-x) - \ln x$, the function $U(t) = v((1+e^t)^{-1})$, and define

$$f_2(t) = (1 + e^{-t})U(t), \tag{7.8}$$

then (7.7) equals (7.2) with $\gamma = \ln\lambda - \ln\mu$. Moreover, in this case the homogeneous equation corresponding to (7.2) is the same as (7.5). Thus, if (7.2) (respectively (7.5)) has a solution then (7.7) (respectively the homogeneous equation corresponding to (7.7)) has a solution as well.

Let $f_2(t)$ be a solution of (7.2) and consider the corresponding function $v(x)$. Applying the Lagrange formula to the difference $v(\lambda x/\,(\lambda x + \mu(1-x))) - v(x)$ on the left-hand side of (7.7) we obtain

$$\begin{aligned} &v\left(\frac{\lambda x}{\lambda x + \mu(1-x)}\right) - v(x) \\ &\quad = v'(\xi)\frac{\varepsilon x(1-x)}{\lambda x + \mu(1-x)}, \qquad \text{for } \xi \in \left[x, \frac{\lambda x}{\lambda x + \mu(1-x)}\right]. \end{aligned}$$

Substituting this expression in (7.7) we get

$$-(1-x) + \varepsilon x(1-x)[v'(\xi) - v'(x)] + a(1-2x)v'(x) = \beta v(x). \tag{7.9}$$

Solving (7.7) with respect to $v'(x)$ and taking into consideration the fact that $a > 0$ and $v(x)$ is a continuously differentiable function, we deduce that $v(x)$ has a smooth second derivative in a neighborhood of the point $x = 1$. Therefore, we apply the Lagrange formula once again to the difference $v'(\xi) - v'(x)$, which appears in the second term of (7.9). So, we obtain

$$a(1-2x)v'(x) = \beta v(x) + 1 - x + O((1-x)^2), \qquad \text{as } x \to 1. \tag{7.10}$$

We now use the logarithmic scale $t = \ln(1-x) - \ln x$ and $U(t)$. Then (7.10) takes the following form

$$\begin{aligned} 2aU'(t)\sinh t = &-\beta U(t) - (1 + e^{-t})^{-1} \\ &+ O((1 + e^{-t})^{-2}), \qquad \text{as } t \to -\infty. \end{aligned} \tag{7.11}$$

The solution of the last equation is

$$
\begin{aligned}
U(t) = {} & \exp\left(-\frac{\beta}{2a}\int_{-\infty}^{t}\frac{du}{\sinh u}\right) \\
& \times\left[C_1 - \int_{-\infty}^{t}\frac{\exp(\beta/2a\int_{-\infty}^{u} ds/\sinh s)}{2a(1+e^{-u})\sinh u}\,du\right. \\
& \left. + \int_{-\infty}^{t}\exp\left(\frac{\beta}{2a}\int_{-\infty}^{u}\frac{ds}{\sinh s}\right)O((1+e^{-u})^{-2})\frac{du}{\sinh u}\right] \\
= {} & \left(\frac{e^{-t}-1}{e^{-t}+1}\right)^{-\beta/2a}\left[C_1 + a^{-1}\int_{-\infty}^{t}(e^{-u}-1\Big)^{(\beta/2a)-1}(e^{-u}+1)^{-(\beta/2a)-2}e^{-u}\,du\right. \\
& \left. + \int_{-\infty}^{t}\left(\frac{e^{-u}-1}{e^{-u}+1}\right)^{\beta/2a}\frac{e^{-u}O((1+e^{-u})^{-2})}{(e^{-u}-1)(e^{-u}+1)}\,du\right], \qquad (7.12)
\end{aligned}
$$

$$
C_1 = \lim_{t\to-\infty} U(t), \qquad t\to-\infty.
$$

The first integral in (7.12) equals

$$
\beta^{-1}(2+\beta/a)^{-1}\left\{1-\left(\frac{e^{-t}-1}{e^{-t}+1}\right)^{\beta/2a}[1+(\beta/a)(1+e^{-t})^{-1}]\right\}.
$$

In view of the inequalities

$$
\frac{e^{-u}-1}{e^{-u}+1} < 1 \quad \text{and} \quad \frac{1}{e^{-u}-1} < \frac{2}{e^{-u}+1}
$$

which hold for $u<0$ the second integral can be written as

$$
\int_{-\infty}^{t}(1+e^{-u})^{-2}O((1+e^{-u})^{-2})d(1+e^{-u}) = O((1+e^{-u})^{-3}).
$$

Substituting these expressions in (7.12) we obtain the following asymptotic formula for $U(t)$

$$
U(t) = C\zeta_1(t) + \zeta_2(t) + O((1+e^{-t})^{-3}), \qquad t\to-\infty, \qquad (7.13)
$$

where

$$\zeta_1(t) = \left(\frac{e^{-t}-1}{e^{-t}+1}\right)^{-\beta/2a},$$

$$\zeta_2(t) = -\frac{1+(\beta/a)(1+e^{-t})^{-1}}{\beta(2+\beta/a)},$$

and

$$C = C_1 + \beta^{-1}(2+\beta/a)^{-1}.$$

Applying Newton's formula to $\zeta_1(t)$ we get

$$\begin{aligned}\zeta_1(t) &= \left(\frac{e^{-t}-1}{e^{-t}+1}\right)^{-\beta/2a}\\ &= \left(1-\frac{2}{1+e^{-t}}\right)^{-\beta/2a}\\ &= 1+(\beta/a)(1+e^{-t})^{-1} + o((1+e^{-t})^{-1}), \qquad t\to-\infty. \end{aligned} \tag{7.14}$$

It follows from (7.13) and (7.14) that

$$U(t) = C_1[1+(\beta/a)(1+e^{-t})^{-1}] + o((1+e^{-t})^{-1}), \qquad t\to-\infty. \tag{7.15}$$

Note that $\zeta_1(t)$ is a solution of the homogeneous equation corresponding to (7.11). This homogeneous equation arises from (7.5) and the homogeneous equation corresponding to (7.7). Now both claims of the theorem follow from (7.8), (7.14), (7.15) and the remark after formula (7.8). □

As follows from (7.15), $\zeta_2(t)$ plays an important role in (7.13). Namely, it ensures that $C_1 = \lim_{t\to-\infty} U(t)$ holds in (7.15). It is remarkable that $f_2(t) = (1+e^{-t})\zeta_2(t)$ is a global solution (that is, a solution on the entire real line) of (7.2).

Theorem 7.2. The functions

$$f_1(t) = -\frac{1+\beta/a+e^{t}}{\beta(2+\beta/a)}, \tag{7.16}$$

$$f_2(t) = -\frac{1+\beta/a+e^{-t}}{\beta(1+\beta/a)}, \tag{7.17}$$

are global solutions of Eqs. (7.1) and (7.2), respectively.

Proof. To prove the theorem it is enough to substitute $f_1(t)$ and $f_2(t)$ from (7.16) and (7.17) into (7.1) and (7.2). $\square$

Note that the functions from (7.16) and (7.17) (which are the equations displayed in Theorem 7.2) do not satisfy the system (7.1)–(7.3) because (7.3) is not fulfilled. On the other hand, we could multiply the right-hand side of (7.17) by e^t and try to solve Eq. (7.1) on $(-\infty, 0]$ with an initial condition $-(1+(1+\beta/a)e^t)/\beta(2+\beta/a)$ given on $[-\gamma, 0]$. However, in this case we will not be able to escape the singularity that (7.1) has at the point $t = \ln(\varepsilon/2a + \sqrt{1+(\varepsilon/2a)^2})$. To overcome these difficulties we need full characterization of the set of all solutions of Eq. (7.2) on the negative half-line.

Denote by $V(t,s)$ and $U(t,s)$ the fundamental functions of Eqs. (7.1) and (7.2), respectively. So, for any fixed s, $V(t,s)$ (respectively $U(t,s)$) is a solution of the homogeneous equation corresponding to (7.1) (respectively (7.2)) with an initial condition given on $[s-\gamma, s]$ by the formula $\varphi(t) = 0$ if $s-\gamma < t < s$ and $\varphi(t) = 1$ if $t = s$. Then $V(s,s) = U(s,s) = 1$ and $V(t,s) = U(t,s) = 0$ for $t < s$. If $0 \le t-s \le \gamma$ then in order to calculate $V(t,s)$ (respectively $U(t,s)$) one has to solve the following ordinary differential equation: $f'(t) = (\varepsilon - 2a\sinh t)^{-1}\,(\lambda + a + \beta - ae^t)f(t)$ (respectively $f'(t) = (\varepsilon - 2a\sinh t)^{-1}(\mu + a + \beta - ae^{-t})f(t)$), $s \le t \le s+\gamma$, with an initial condition $f(s) = 1$. Solving these equations we obtain the following expressions for the functions $V(t,s)$ and $U(t,s)$

$$V(t,s) = \left(\frac{1+x_2 e^s}{1+x_2 e^t}\right)^b \left(\frac{1+x_1 e^s}{1+x_1 e^t}\right)^c, \tag{7.18}$$

$$U(t,s) = e^{s-t}\left(\frac{1+x_2 e^s}{1+x_2 e^t}\right)^b \left(\frac{1+x_1 e^s}{1+x_1 e^t}\right)^c, \tag{7.19}$$

where $x_{1,2} = \varepsilon/2a \pm \sqrt{1+(\varepsilon/2a)^2}$ are the roots of the equation $x^2 - (\varepsilon/a)x - 1 = 0$,

$$b = \frac{1+(\mu+\beta)/a + x_2}{x_1 - x_2}, \qquad c = -\frac{1+(\mu+\beta)/a + x_1}{x_1 - x_2}. \tag{7.20}$$

It is easy to verify that $b > 0$ and the identity $b + c = -1$ holds. It follows from (7.18) and (7.19) that

$$U(t,s) = e^{s-t}V(t,s), \qquad s \le t \le s+\gamma. \tag{7.21}$$

Since $U(t,s) = V(t,s) = 0$ if $t < s$, (7.21) is obviously fulfilled for all $t < s$. If $t > s+\gamma$ then the formulas for $V(t,s)$ and $U(t,s)$ are more complicated.

Nevertheless, it turns out that (7.21) holds in this case as well. Consider the equations

$$f'(t) = (\varepsilon - 2a\sinh t)^{-1}\{\lambda[f(t) - f(t-\gamma)] + (a+\beta - a\,e^{t})f(t)\}, \quad (7.22)$$

$$f'(t) = (\varepsilon - 2a\sinh t)^{-1}\{\mu[f(t) - f(t-\gamma)] + (a+\beta - a\,e^{-t})f(t)\} \quad (7.23)$$

and denote by $f_1(\sigma,\varphi)(t)$ (respectively $f_2(\sigma,\varphi)(t)$), $\sigma \in \mathbb{R}$, $\varphi \in C([\sigma-\gamma,\sigma])$, the solution of (7.22) (respectively (7.23)) on $[\sigma,\infty)$ with an initial condition φ.

Theorem 7.3.

(i) For every real σ and $\varphi \in C([\sigma-\gamma,\sigma])$

$$f_1(\sigma, e^{(\cdot)}\varphi)(t) = e^{t} f_2(\sigma,\varphi)(t), \qquad t \geq \sigma. \quad (7.24)$$

(ii) The identity (7.21) is fulfilled for all real t and s.

Proof. From Hale (1977, Thm 6.3.2) we obtain the following representation for the functions $f_1(\sigma, e^{(\cdot)}\varphi)(t)$ and $f_2(\sigma,\varphi)(t)$ for $t \geq \sigma$

$$\begin{aligned} f_2(\sigma,\varphi)(t) =& U(t,\sigma)\varphi(\sigma) \\ & - \mu \int_{\sigma-\gamma}^{\sigma} \varphi(s) U(t, s+\gamma)[\varepsilon - 2a\sinh(s+\gamma)]^{-1}\, ds, \end{aligned} \quad (7.25)$$

$$\begin{aligned} f_1(\sigma, e^{(\cdot)}\varphi)(t) =& V(t,\sigma) e^{\sigma}\varphi(\sigma) \\ & - \lambda \int_{\sigma-\gamma}^{\sigma} \varphi(s)\, e^{s} V(t, s+\gamma)[\varepsilon - 2a\sinh(s+\gamma)]^{-1}\, ds. \end{aligned} \quad (7.26)$$

First, we shall prove claim (i) in the case of $\sigma \leq t \leq \sigma+\gamma$. Since $t \leq s+2\gamma$ for all $s \in [\sigma-\gamma,\sigma]$ one can apply (7.21) to both $V(t,\sigma)$ and $V(t,s+\gamma)$ in the right-hand side of (7.26). Making simple calculations we get (7.24). Now, we are in a position to prove (ii). Let us fix s and divide the interval $[s,\infty)$ into segments of length $\gamma : [s,\infty) = \bigcup_{n\geq 0}[s+n\gamma, s+(n+1)\gamma)$. We shall prove by induction that (7.21) holds in every segment $[s+n\gamma, s+(n+1)\gamma)$, $n = 0, 1, \ldots$. For $n = 0$ the claim has already been proved. Assuming that (ii) holds true for $t \in [s+k\gamma, s+(k+1)\gamma)$ and some integer $k \geq 0$ we shall prove that (7.21) is satisfied in the next segment as well. Indeed, to find the functions $V(t,s)$ and $U(t,s)$ on $[s+(k+1)\gamma, s+(k+2)\gamma)$ one must solve Eqs. (7.22) and (7.23), respectively, taking as initial conditions the already calculated values of these functions on $[s + k\gamma, s + (k+1)\gamma)$. According to the induction hypothesis, (7.21) holds on $[s + k\gamma, s + (k+1)\gamma)$. Applying (i) with $\sigma = s + (k+1)\gamma$, $\varphi(\cdot) = e^{-s}U(\cdot, s)$ and $t \in [s + (k+1)\gamma, s + (k+2)\gamma)$ we get the claim for $n = k + 1$. Thus, (ii) is proved.

In order to prove (7.24) for $\sigma \le t \le \sigma + \gamma$ we have used only formula (7.21) with $t - s \le \gamma$. Since, according to (ii) the last formula holds for all t and s it follows that (7.24) is fulfilled for all $t \ge \sigma$. □

Consider the following domain in the plane

$$(s, t) : D = \{(s, t) : s \le t \le 0\}.$$

Theorem 7.4. The function $U(t, s)$ is continuous and bounded in D.

Proof. The coefficients of Eq. (7.23) are bounded continuous functions provided $t \le 0$. For any fixed $s \le 0$, the function $U(t, s)$ is given by formula (7.19) if $s \le t \le s + \gamma$ and $U(t, s)$ satisfies (7.23) with an initial condition $f_{s+\gamma}(\cdot) = U_{s+\gamma}(\cdot, s)$ if $t \ge s + \gamma$. Here, as it is generally assumed in the theory of functional-differential equations, for any $f \in C([s, t]\}, t \ge s + \gamma$ and $\tau \in [s + \gamma, t]$, $f_\tau(\cdot)$ denotes a function belonging to $C([-\gamma, 0])$ which is defined by $f_\tau(\theta) = f(\tau + \theta), \theta \in [-\gamma, 0]$. In view of (7.19), since $s + \gamma + \theta \in [s, s + \gamma]$ provided $\theta \in [-\gamma, 0]$ it follows that

$$U_{s+\gamma}(\theta, s) = e^{-\theta-\gamma}\left(\frac{1 + x_2\, e^{s}}{1 + x_2\, e^{s+\gamma+\theta}}\right)^{b}\left(\frac{1 + x_1\, e^{s}}{1 + x_1\, e^{s+\gamma+\theta}}\right)^{c}, \qquad \theta \in [-\gamma, 0]. \quad (7.27)$$

The last function is continuous in both θ and s if $\theta \in [-\gamma, 0]$ and $s \le -\gamma$. Being a solution of (7.23) with the continuous initial condition (7.27), $U(t, s)$ is continuous with respect to t if $t \ge s + \gamma$. The continuity of $U(t, s)$ for $t \in [s, s + \gamma]$ follows from (7.19). It remains to prove that $U(t, s)$ is continuous with respect to s for every fixed $t \le 0$. The case $s \ge t - \gamma$ holds by inspection. The proof is not trivial if $s < t - \gamma$. Let $s < t - \gamma$ and $\{s_n\}$ be a sequence converging to s. Then, because of the continuity of (7.27) with respect to both θ and s, the corresponding sequence of initial conditions $\{U_{s_n+\gamma}(\cdot, s_n)\}$ will converge uniformly to $U_{s+\gamma}(\cdot, s)$. Thus, the continuity of $U(t, s)$ in s follows from the theorem for the continuous dependence of the solutions of functional differential equations on the initial data, (Hale (1977, Thm 2.2.2).

In order to prove the second claim of the theorem let us rewrite (7.23) in the form

$$f'(t) = -A(t)f(t) - B(t)f(t-\gamma). \tag{7.28}$$

where $A(t) = (\varepsilon - 2a\sinh t)^{-1}(a e^{-t} - a - \mu - \beta)$, $B(t) = \mu(\varepsilon - 2a\sinh t)^{-1}$ and set $\alpha = -\ln(1 + (\beta + 2\mu)/a)$. It is easy to see that the coefficient $A(t)$ is a decreasing function of t, for $t < 0$, whereas $B(t)$ is an increasing function of t, and for $t < \alpha$ the inequality $B(t) < A(t)$ holds.

Consider the following domains in the plane (s, t)

$$D_1 = \{(s,t) \in D : s \le t \le \alpha\},$$

$$D_2 = \{(s,t) \in D : t \ge s \ge \alpha\},$$

$$D_3 = \{(s,t) \in D : t \ge \alpha, s \le \alpha\}.$$

Obviously, $D = D_1 \cup D_2 \cup D_3$ and it is enough to prove the claim in each of domains D_1, D_2, and D_3. Since D_2 is a compact set and $U(t,s)$ is a continuous function the proof is trivial in D_2.

For D_1 consider the following subdomains of $D_1 : D_1^n = \{(s,t) \in D_1 : t \le \alpha - 1/n\}$, $n = 1, 2, \ldots$. Since the closure of the set $\bigcup_{n\ge1} D_1^n$ coincides with D_1 it follows from the continuity of $U(t,s)$ that it is sufficient to prove its boundedness in each of domains $D_1^n, n = 1, 2, \ldots$. If $t \le \alpha - 1/n$ then $A(t) \ge A(\alpha - 1/n)$, and $\sup_{t\le\alpha-1/n} |B(t)| \le B(\alpha - 1/n) < A(\alpha - 1/n)$ and therefore, according to Hale (1977, Eq. (5.9.2)), the trivial solution of (7.28) is uniformly stable for $t \le \alpha - 1/n$. This means that for any $\sigma \le \alpha - 1/n$ and $d > 0$ there exists $\delta = \delta(d)$ such that the inequality $\|\varphi\| < \delta$ implies $\|(f_2)_t(\sigma,\varphi)\| < d$ for all $\sigma \le t \le \alpha - 1/n$. Here $\|\cdot\|$ denotes a sup-norm and $f_2(\sigma,\varphi)$ is the same as in Theorem 7.3.

Let us now represent Eq. (7.28) in operator form. That is

$$f'(t) = L(t, f_t(\cdot)),$$

where $L(t,\varphi), t \le 0, \varphi \in C([-\gamma, 0])$ is the following operator

$$L(t,\varphi) = -A(t)\varphi(0) - B(t)\varphi(-\gamma).$$

Because if $t < \alpha - 1/n$, then both $A(t)$ and $B(t)$ are between 0 and 1, we have that

$$|L(t,\varphi)| \le m\|\varphi\|, \qquad \text{with } m = 2. \tag{7.29}$$

We apply Lemma 6.6.2. in Hale (1977) to (7.28) and use (7.29) to obtain the claim in D_1^n.

For any fixed $s < \alpha$, the function $U(t, s)$ satisfies

$$f(t) = \int_{\alpha}^{t} L(u, f_u(\cdot))\, du + U(\alpha, s), \tag{7.30}$$

on $[\alpha, 0]$ with the initial condition $f_\alpha(\cdot) = U_\alpha(\cdot, s)$. Note that if $t < 0$ then (7.29) also holds with $m = (2\mu + \beta)/\varepsilon$. Since $(s, \alpha) \in D_1$ it follows from (7.29) and (7.30) that

$$\begin{aligned}|U(t, s)| &\leq \int_{\alpha}^{t} |L(u, U_u(\cdot, s)\, du| + |U(\alpha, s)| \\ &\leq m \int_{\alpha}^{t} \| U_u(\cdot, s) \|\, du + K,\end{aligned}$$

where $K = \sup_{(t,s)\in D_1} |U(t, s)|$. Therefore

$$\| U_t(\cdot, s)\| \leq m \int_{\alpha}^{t} \| U_u(\cdot, s)\|\ du + K. \tag{7.31}$$

Applying Gronwall's lemma to (7.31) we get

$$\| U_t(\cdot, s) \| \leq K_1$$

where $K_1 = Ke^{m\alpha}$, $t \in [\alpha, 0]$. Hence, $\sup_{\alpha \leq t \leq 0} |U(t, s)| \leq K_1$. Since the constant K_1 does not depend on s it follows that $U(t, s)$ is bounded in D_3 as well. □

Remark. All results cited in the proof relate to the case when the functional-differential equation is given on the whole real line, whereas we consider it in the interval $(-\infty, \alpha - 1/n]$. However, setting in (7.27) $A(t) = A(\alpha - 1/n)$, $B(t) = B(\alpha - 1/n)$, for $t > \alpha - 1/n$ we get an equation that is defined on the whole real line coinciding with (7.27) in $(-\infty, \alpha - 1/n]$. Since all cited results hold true for the last equation it follows that they hold for (7.27) in the interval $(-\infty, \alpha - 1/n]$ as well.

We now return to the problem of the explicit form of the solution of the system of functional-differential Eqs. (7.1)–(7.3).

We begin this section with the following lemma.

Lemma 7.5. Let us set $X(\sigma, t) = f_2(\sigma, 1 + \beta/a + e^{-(\cdot)})(t)$, $\sigma \leq -\gamma$, where as in Theorem 7.3, $f_2(\sigma, 1 + \beta/a + e^{-(\cdot)})(t)$ denotes the solution of Eq. (7.23) with an initial condition $\varphi(t) = 1 + \beta/a + e^{-t}$ given on $[\sigma - \gamma, \sigma]$. Then

$$\text{(i)}\ \ X(\sigma, t) > 1 + b/a + e^{-t}, \qquad \text{for } t \in [\sigma, \sigma + \gamma], \tag{7.32}$$

$$\text{(ii)}\ \ \max_{t\in[\sigma,\sigma+\gamma]} [X(\sigma, t) - 1 - beta/a - e^{-t}]$$
$$= O(e^{\sigma}), \qquad \text{as } \sigma \to -\infty. \tag{7.33}$$

Proof. If $t \in [\sigma, \sigma + \gamma]$ then $X(\sigma, t)$ is calculated from (7.25), where $U(t, s)$ is given by (7.19). We first set $\varphi(s) = 1$ in (7.25) and evaluate the integral to obtain

$$K_1(t) = e^{\sigma - t}\left(\frac{1 + x_2 e^{\sigma}}{1 + x_2 e^{t}}\right)^{b}\left(\frac{1 + x_1 e^{\sigma}}{1 + x_1 e^{t}}\right)^{c}$$
$$+ \frac{\mu}{am}\left[e^{-t} + 1 + \frac{\lambda + \beta}{a} - e^{\sigma - t}\left(\frac{1 + x_2 e^{\sigma}}{1 + x_2 e^{t}}\right)^{b}\left(\frac{1 + x_1 e^{\sigma}}{1 + x_1 e^{t}}\right)^{c}\right.$$
$$\left.\times\left(e^{-\sigma} + 1 + \frac{\lambda + \beta}{a}\right)\right]. \tag{7.34}$$

We next set $\varphi(s) = e^{-s}$ in (7.25) and evaluate the integral to obtain

$$K_2(t) = e^{-t}\left(\frac{1 + x_2 e^{\sigma}}{1 + x_2 e^{t}}\right)^{b}\left(\frac{1 + x_1 e^{\sigma}}{1 + x_1 e^{t}}\right)^{c}$$
$$+ \frac{\lambda}{am}\left\{1 + \left(1 + \frac{\mu + \beta}{a}\right)e^{-t} - e^{\sigma - t}\left(\frac{1 + x_2 e^{\sigma}}{1 + x_2 e^{t}}\right)^{b}\left(\frac{1 + x_1 e^{\sigma}}{1 + x_1 e^{t}}\right)^{c}\right.$$
$$\left.\times\left[1 + \left(1 + \frac{\mu + \beta}{a}\right)e^{-\sigma}\right]\right\}. \tag{7.35}$$

In (7.34) and (7.35)

$$m = -(x_1 - x_2)^2 bc = \left(1 + \frac{\mu + \beta}{a}\right)\left(1 + \frac{\lambda + \beta}{a}\right) - 1. \tag{7.36}$$

Note that $X(\sigma, t) - 1 - \beta/a - e^{-t}$ equals $(1 + \beta/a)K_1(t) + K_2(t) - 1 - \beta/a - e^{-t}$, so that the derivative of $X(\sigma, t) - 1 - \beta/a - e^{-t}$ with respect to t is

$$
\begin{aligned}
- m^{-1}&\left[\frac{\lambda}{a}\left(1 + \frac{\mu + \beta}{a}\right) + \frac{\mu}{a}(1 + \beta/a) - m\right] \\
&\times \left\{1 + \frac{e^{t}[1 + (\mu + \beta)/a] - 1}{(1 + x_1 e^{t})(1 + x_2 e^{t})}\left(\frac{1 + x_2 e^{\sigma}}{1 + x_2 e^{t}}\right)^{b}\right. \\
&\times \left.\left(\frac{1 + x_1 e^{\sigma}}{1 + x_1 e^{t}}\right)^{c}[e^{\sigma}(1 + (\lambda + \beta)/a) + 1]\right\} e^{-t}.
\end{aligned}
$$

It is easy to verify that $(\lambda/a)(1 + (\mu + \beta)/a) + (\mu/a)(1 + \beta/a)$ is less than m. So, the first factor in this expression is negative. Simple calculations show that the second factor is an increasing function of t. Thus, the second factor reaches the minimum value $m e^{2\sigma}(1 + x_1 e^{\sigma})^{-1}(1 + x_2 e^{\sigma})^{-1} > 0$ at the point $t = \sigma$. Therefore, the function $X(\sigma, t) - 1 - \beta/a - e^{-t}$ increases on $[\sigma, \sigma + \gamma]$ and $X(\sigma, \sigma) - 1 - \beta/a - e^{-\sigma} = 0$, so (7.32) holds. Further, $X(\sigma, t) - 1 - \beta/a - e^{-t}$ reaches its maximum at $t = \sigma + \gamma$. We set $t = \sigma + \gamma$ in (7.34) and define a new function $\bar{K}_1(u)$ so that $\bar{K}_1(e^{\sigma}) = K_1(\sigma + \gamma)$. Then

$$
\bar{K}_1(u) = \frac{\mu}{am} \cdot \frac{\mu}{\gamma} R(u) + S(u), \tag{7.37}
$$

where

$$
R(u) = u^{-1}\left[1 - \left(\mu \frac{1 + x_2 u}{\mu + \lambda x_2 u}\right)^{b}\left(\mu \frac{1 + x_1 u}{\mu + \lambda x_1 u}\right)^{c}\right], \tag{7.38}
$$

$$
\begin{aligned}
S(u) = &\frac{\mu}{\lambda}\left(\mu \frac{1 + x_2 u}{\mu + \lambda x_2 u}\right)^{b}\left(\mu \frac{1 + x_1 u}{\mu + \lambda x_1 u}\right)^{c} \\
&\times \left[1 - \frac{\mu}{am}\left(1 + \frac{\lambda + \beta}{a}\right)\right] + \frac{\mu}{am}\left(1 + \frac{\lambda + \beta}{a}\right).
\end{aligned} \tag{7.39}
$$

In a neighborhood of $u = 0$,

$$
\left(\mu \frac{1 + x_2 u}{\mu + \lambda x_2 u}\right)^{b}\left(\mu \frac{1 + x_1 u}{\mu + \lambda x_1 u}\right)^{c} = 1 + \frac{\varepsilon}{\mu}\left(1 + \frac{\lambda + \beta}{a}\right)u + O(u^2), \tag{7.40}
$$

and hence

$$R(u) = -\frac{\varepsilon}{\mu}\left(1 + \frac{\lambda + \beta}{a}\right) + O(u), \qquad \text{as } u \to 0. \tag{7.41}$$

Taking in (7.39) only the first term of the Taylor series of the function in the left-hand side of (7.40) we get

$$\begin{aligned} S(u) = &\frac{\mu}{\lambda}\left[1 - \frac{\mu}{am}\left(1 + \frac{\lambda + \beta}{a}\right)\right] \\ &+ \frac{\mu}{am}\left(1 + \frac{\lambda + \beta}{a}\right) + O(u), \qquad \text{as } u \to 0. \end{aligned} \tag{7.42}$$

It follows from (7.37), (7.41), and (7.42) that

$$\bar{K}_1(u) = \mu/\lambda + O(u), \qquad \text{as } u \to 0. \tag{7.43}$$

Similarly, if we set $u = e^{\sigma}$ and evaluate $K_2(t) - e^{-t}$ at $t = \sigma + \gamma$ we obtain

$$\bar{K}_2(u) - \frac{\mu}{\lambda}u^{-1} = R(u)\left[\frac{\lambda}{am}\left(u + 1 + \frac{\mu + \beta}{a}\right) - 1\right] + \frac{\varepsilon\lambda}{a\mu m},$$

where $\bar{K}_2(u)$ is defined by $\bar{K}_2(e^{\sigma}) = K_2(\sigma + \gamma)$. From (7.41) we get

$$\bar{K}_2(u) - \frac{\mu}{\lambda}u^{-1} = \frac{\varepsilon}{\mu}\left(1 + \frac{\beta}{a}\right) + O(u), \qquad \text{as } u \to 0. \tag{7.44}$$

Because $X(\sigma, t) - 1 - \beta/a - e^{-t} = K_2(t) - e^{-t} + (1 + \beta/a)[K_1(t) - 1]$ the second claim of the lemma follows from (7.43) and (7.44). $\square$

Theorem 7.6. Equation (7.5) has a non-trivial solution on the negative half-line.

Proof. Consider the following functions defined on $(-\infty, 0]$:

$$\eta_n(t) = \begin{cases} 1 + \beta/a + e^{-t} & t \leq -n\gamma \\ X(-n\gamma, t) & t \in [-n\gamma, 0]. \end{cases}$$

We shall prove that the sequence $\{\eta_n\}_{n\geq 1}$, converges uniformly to some function $\eta(t)$ that satisfies (7.5) on the whole negative half-line.

Consider the series

$$\eta_1(t) + \sum_{n=1}^{\infty} [\eta_{n+1}(t) - \eta_n(t)]. \tag{7.45}$$

From (7.25) we obtain

$$\eta_{n+1}(t) - \eta_n(t)$$

$$= \begin{cases} 0 & t \le -(n+1)\gamma \\ X(t) - 1 - \beta/a - e^{-t} & t \in [-(n+1)\gamma, -n\gamma] \\ [X(-n\gamma) - 1 - \beta/a - e^{n\gamma}]U(t, -n\gamma) & \\ \quad -\mu \int_{-(n+1)\gamma}^{-n\gamma} [X(s) - 1 - \beta/a - e^{-s}] & \\ \quad \times[\varepsilon - 2a \sinh(s+\gamma)]^{-1} U(t, s+\gamma)\, ds & t \in [-n\gamma, 0], \end{cases}$$

where, for the sake of brevity, we have omitted the first argument of $X(-(n+1)\gamma, t)$. Therefore, it follows from Theorem 7.4, Lemma 7.5, and convergence of the integral $\int_{-\infty}^{0} (\varepsilon - 2a \sinh s)^{-1}\, ds$ that for all n sufficiently large

$$|\eta_{n+1}(t) - \eta_n(t)| \le K\, e^{-n\gamma},$$

where K is a constant. Thus, series (7.45) converges uniformly to $\eta(t)$. It remains to prove that $\eta(t)$ satisfies (7.5) on $(-\infty, 0]$.

Let $t \in [-\gamma, 0]$. For every $n \ge 1$, the function $\eta_n(t)$ satisfies (7.5) on $[-\gamma, 0]$, so it follows that

$$\eta_n' = -A(t)\eta_n(t) - B(t)\eta_n(t-\gamma), \qquad \text{for } n = 1, 2, \ldots, \tag{7.46}$$

where $A(t)$ and $B(t)$ are the coefficients defined after (7.28). From (7.46) we deduce that

$$\begin{aligned} &\eta_1' + \sum_{n=1}^{\infty} [\eta_{n+1}'(t) - \eta_n'(t)] \\ &\quad = -A(t)\left\{\eta_1(t) + \sum_{n=1}^{\infty} [\eta_{n+1}(t) - \eta_n(t)]\right\} \\ &\qquad - B(t)\left\{\eta_1(t-\gamma) + \sum_{n=1}^{\infty} [\eta_{n+1}(t-\gamma) - \eta_n(t-\gamma)]\right\} \\ &\quad = -A(t)\eta(t) - B(t)\eta(t-\gamma). \end{aligned}$$

Therefore, differentiating series (7.45) we obtain that in this case $\eta(t)$ satisfies Eq. (7.5).

If $t \in [-(k+1)\gamma, -k\gamma]$, for $k \geq 1$, then we represent $\eta(t)$ in the form

$$\eta(t) = \eta_{k+1}(t) + \sum_{n=k+1}^{\infty} [\eta_{n+1}(t) - \eta_n(t)]$$

and repeat the same argument for (7.46) with $n = k+1, k+2, \ldots.$ □

Now, we are able to characterize the set of all solutions of (7.2) on the negative half-line.

Theorem 7.7. The function

$$f_2(t) = -\frac{1 + \beta/a + e^{-t}}{\beta(2 + \beta/a)} + C\eta(t), \tag{7.47}$$

where C, a generic constant, is a general solution of Eq. (7.2) on $(-\infty, 0]$.

Proof. According to Theorems 7.2 and 7.6 $f_2(t)$ is a solution of Eq. (7.2) for all C. It remains to prove that every solution of this equation can be represented in the form (7.47) with some constant C. Let $f(t)$ be such a solution. Then, from Theorem 7.1, $f(t) = C_1(1 + \beta/a + e^{-t}) + o(1)$ as $t \to -\infty$ with some constant C_1. In addition, from the proof of Theorem 7.6 $\eta(t) = 1 + \beta/a + e^{-t} + o(1)$ as $t \to -\infty$. Therefore, if we set C equal to $C_1 + \beta^{-1}(2 + \beta/a)^{-1}$ in (7.47), then we obtain a solution $f_2(t)$ that has the same asymptotic behavior as $f(t)$ for $t \to -\infty$. Thus, for every $\delta > 0$ there exists a $\sigma < 0$ such that

$$\sup_{t \leq \sigma} |f(t) - f_2(t)| < \delta. \tag{7.48}$$

If $t \in [\sigma, 0]$ then, from Hale (1977, Thm 6.3.2), $f(t)$ has the following representation

$$\begin{aligned} f(t) = {} & f(\sigma)U(t,\sigma) - \mu \int_{\sigma-\gamma}^{\sigma} f(s)U(t, s+\gamma)[\varepsilon - 2a\sinh(s+\gamma)]^{-1}\, ds \\ & + \int_{\sigma}^{t} U(t,s)(\varepsilon - 2a\sinh s)^{-1}\, ds, \end{aligned}$$

and that the representation for $f_2(t)$ is obtained by replacing $f(t)$ with $f_2(t)$. Hence,

$$f(t) - f_2(t) = [f(\sigma) - f_2(\sigma)]U(t, \sigma) - \mu \int_{\sigma-\gamma}^{\sigma} [f(s) - f_2(s)]U(t, s+\gamma)[\varepsilon - 2a\sinh(s+\gamma)]^{-1}\, ds.$$

From (7.48) and the boundedness of $U(t, s)$ we get

$$\max_{t\in[\sigma,0]} |f(t) - f_2(t)| < K\delta, \qquad K = \text{const.} \tag{7.49}$$

where K is a constant. Because $\delta > 0$ is arbitrary in (7.48) and (7.49) it follows that $f(t) = f_2(t)$. □

From the last theorem we are in a position to prove our main result about the solution of system (7.1)–(7.3).

Theorem 7.8. The system (7.1)–(7.3) has a unique solution, provided that μ/a is small enough.

Proof. According to Theorem 7.7 the general solution of (7.2) on $(-\infty, 0]$ is given by (7.47). To satisfy (7.1) and (7.3) one must multiply (7.47) by e^t on $[-\gamma, 0]$ and solve (7.1) with an initial condition $e^t f_2(t)$, for $t \in [-\gamma, 0]$. We shall show that taking a suitable constant C in (7.47) we can escape the singularity that (7.1) has at $t = \ln x_1$. Then, because the coefficients of (7.1) are unbounded only in a neighborhood of $t = \ln x_1$, the existence and uniqueness of the solution of (7.1) follow from Hale (1977, Thm 6.1.1).

We set $k = \ln x_1$ and take an arbitrary $\delta \in (0, \min(k, \gamma/2))$. From (7.18) the solution of (7.1) has the following representation in $[k-\delta, k+\delta]$:

$$\begin{aligned} f_1(t) &= V(t, k-\delta) f_1(k-\delta) \\ &\quad - \lambda \int_{k-\delta-\gamma}^{k-\delta} f_1(s)[\varepsilon - 2a\sinh(s+\gamma)]^{-1} V(t, s+\gamma)\, ds \\ &\quad + \int_{k-\delta}^{t} (\varepsilon - 2a\sinh s)^{-1} V(t, s)\, ds \\ &= (1 + x_2 e^t)^{-b}(1 + x_1 e^t)^{-c} \\ &\quad \times \Big\{ (1 + x_2 e^{k-\delta})^b (1 + x_1 e^{k-\delta})^c f_1(k-\delta) + \int_{k-\delta}^{t} [1 - \lambda f_1(s-\gamma)] \\ &\quad \times (\varepsilon - 2a\sinh s)^{-1}(1 + x_2 e^s)^b (1 + x_1 e^s)^c\, ds \Big\}. \end{aligned} \tag{7.50}$$

Because $1 + x_2 e^k = 0$ and $b > 0$, the function $f_1(t)$ is bounded in a neighborhood of $t = k$ only if

$$\begin{aligned} 0 =& (1 + x_2 e^{k-\delta})^b (1 + x_1 e^{k-\delta})^c f_1(k-\delta) \\ &+ \int_{k-\delta}^{t} [1 - \lambda f_1(s-\gamma)](\varepsilon - 2a \sinh s)^{-1} (1 + x_2 e^s)^b (1 + x_1 e^s)^c \, ds. \end{aligned} \tag{7.51}$$

From l'Hôpital's rule one can verify that (7.51) is sufficient for boundedness of $f_1(\cdot)$. From Theorem 7.7 if $t \in [k-\delta-\gamma, k-\delta]$, then

$$f_1(t) = f_1^{(1)}(t) + f_1^{(2)}(t) + C f_1^{(3)}(t), \tag{7.52}$$

for some constant C, where $f_1^{(1)}(t)$ is a solution of (7.1) with initial condition $f_1^{(1)}(0) = 0$, $f_1^{(2)}(t)$ is a solution of (7.22) with initial condition on $[-\gamma, 0]$ equal to $-(1 + e^t(1 + (\beta/a)))/(\beta(2 + (\beta a)))$, and $f_1^{(3)}(t)$ is a solution of (7.22) with initial condition on $[-\gamma, 0]$ equal to $e^t \eta(t)$. Note that if $t \le k - \delta$, then the coefficients of (7.1) and (7.22) are bounded and, therefore, all functions in (7.52) are well defined.

Substituting $f_1(t)$ from (7.52) into (7.51) we obtain a linear algebraic equation, $\tilde{A}C + \tilde{B} = 0$, for the constant C that always has a solution $C = \bar{C}$ if $\tilde{A} \neq 0$. If $C \neq 0$, then setting $C = \bar{C}$ in (7.52) ensures that the condition (7.51) holds. Therefore we obtain a bounded solution of (7.1) in $[k-\delta, k+\delta]$ given by (7.50). Because the coefficients of (7.1) are bounded continuous functions for $t \ge k + \delta$, it follows that the solution can be extended to the entire positive half-line.

To complete the proof, it is sufficient to show that

$$\begin{aligned} \tilde{A} =& (1 + x_2 e^{k-\delta})^b (1 + x_1 e^{k-\delta})^c f_1^{(3)}(k-\delta) \\ &- \lambda \int_{k-\delta}^{t} f_1^{(3)}(s-\gamma)(\varepsilon - 2a \sinh s)^{-1} (1 + x_2 e^s)^b (1 + x_1 e^s)^c \, ds \\ &\neq 0. \end{aligned} \tag{7.53}$$

To show this, we assume the contrary. Then, repeating the above argument that was used for $f_1(t)$ and (7.1), we see that

$$\begin{aligned} f_1^{(3)}(t) =& (1 + x_2 e^t)^{-b} (1 + x_1 e^t)^{-c} \\ &\times \Big[(1 + x_2 e^{k-\delta})^b (1 + x_1 e^{k-\delta})^c f_1^{(3)}(k-\delta) \\ &-\lambda \int_{k-\delta}^{t} f_1^{(3)}(s-\gamma)(\varepsilon - 2a \sinh s)^{-1} (1 + x_2 e^s)^b (1 + x_1 e^s)^c \, ds \Big], \end{aligned}$$

remains bounded in $[k-\delta, k+\delta]$. Therefore, $f_1^{(3)}(\cdot)$ can be extended on $[k+\delta, \infty)$ to satisfy (7.22). From Theorem 7.3(i) and Theorem 7.6 we conclude that

$$\zeta(t) = \begin{cases} \eta(t), & t \leq 0, \\ e^{-t} f_1^{(3)}(t), & t \geq 0, \end{cases}$$

is a non-trivial global solution of (7.5). Then, it follows from Theorem 7.2 that $F(t) = f_2(t) + \zeta(t)/(\beta(2+\beta/a))$, $f_2(t)$ being given by (7.17), is a global solution of (7.2). Moreover, (7.6) implies that $F(t) = o(1)$, as $t \to -\infty$. Recall that the function $\xi(t)$ from Theorem 3.3 has the same asymptotic as $t \to -\infty$. Utilizing the same arguments as in the proof of Theorem 7.7 we conclude that $F(t) = \xi(t)$, for $t < \ln x_1$. Now, inequality (3.31) and the explicit formulae (3.21) and (3.22) for $d(t)$ and $D(t)$ imply that $F(t)$ has a singularity at point $t = \ln x_1$. Thus, it cannot be a global solution of (7.2). □

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